

Conditional decoupling of random interlacements

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ABSTRACT. We prove a conditional decoupling inequality for the model of random interlacements in dimension $d \geq 3$: the conditional law of random interlacements on a box (or a ball) A_1 given the (not very “bad”) configuration on a “distant” set A_2 does not differ a lot from the unconditional law. The main method we use is a suitable modification of the *soft local time* method of [13], that allows dealing with conditional probabilities.

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1. INTRODUCTION

Random interlacements were introduced by Sznitman in [17], to model the trace of the simple random walk on the discrete torus $\mathbb{Z}_n^d := \mathbb{Z}^d / n\mathbb{Z}^d$ or the discrete cylinder $\mathbb{Z} \times \mathbb{Z}^{d-1}$, in dimension $d \geq 3$. Detailed treatments and reviews of recent results can be found in the recent books [4, 6, 19]. Loosely speaking, the model of random interlacements in \mathbb{Z}^d , $d \geq 3$, is a stationary Poissonian soup of bi-infinite simple random walk trajectories on the integer lattice. There is a parameter $u > 0$ entering the intensity measure of the Poisson process, the larger u is the more trajectories are thrown in. The sites of \mathbb{Z}^d that are not touched by the trajectories constitute the *vacant set* \mathcal{V}^u , and the union of all trajectories constitutes the interlacement set $\mathcal{I}^u = \mathbb{Z}^d \setminus \mathcal{V}^u$. The random interlacements are constructed simultaneously for all $u > 0$ in such a way that $\mathcal{I}^{u_1} \subset \mathcal{I}^{u_2}$ if $u_1 < u_2$. In fact, the law of the vacant set at level u can be uniquely characterized by the following identity:

$$(1.1) \quad \mathbb{P}[A \subset \mathcal{V}^u] = \exp(-u \operatorname{cap}(A)),$$

where $\operatorname{cap}(A)$ is the *capacity* of a finite set $A \subset \mathbb{Z}^d$. Informally, the capacity measures how “big” is the set from the point of view of the walk, see Section 6.5 of [11] for formal definitions, or Section 2 below.

The model of random interlacements naturally has more independence built in than just one random walk on the torus or the cylinder (because on a fixed set one observes traces of *independent* trajectories). Still, the analysis of random interlacements is difficult because of the long-range dependencies present there. For example, in (1.68) from [17] we can see that

$$(1.2) \quad \operatorname{Cov}(1_{x \in \mathcal{I}^u}, 1_{y \in \mathcal{I}^u}) \sim \frac{c_d u}{\|x - y\|^{d-2}} \quad \text{as} \quad \|x - y\| \rightarrow \infty,$$

which means that the “degree of dependence” decreases polynomially in the distance.

Naturally, one is interested in “decoupling” the events supported on distant regions; that is, to argue that they are approximately independent to a certain degree. One possible approach to quantify that degree is the following: given finite sets $A_1, A_2 \subset \mathbb{Z}^d$ and

functions $f_1 : \{0, 1\}^{A_1} \rightarrow [0, 1]$ and $f_2 : \{0, 1\}^{A_2} \rightarrow [0, 1]$ depending on the interlacements set intersected with A_1 and A_2 respectively, we have

$$(1.3) \quad \text{Cov}_u(f_1, f_2) \leq c_d u \frac{\text{cap}(A_1) \text{cap}(A_2)}{\text{dist}(A_1, A_2)^{d-2}},$$

as proved in formula (2.15) of [17], see also (8.1.1) in [6]. However, the polynomial error term in (1.3) can complicate one's life in many applications (and, e.g. in the case when the diameters of these sets are of the same order as the distance between them, (1.3) is simply of no use); on the other hand, while (1.3) can be improved to some degree [2], the error term there should always be at least polynomial, as (1.2) shows. To circumvent this difficulty, one first may note that usually the “interesting” events/functions are *monotone* (i.e., increasing or decreasing). For e.g. increasing events, we know that their probabilities increase as the parameter u increases. Note also that the FKG inequality (see [21], Theorem 3.1) gives us

$$(1.4) \quad \mathbb{E}^u[g_1 g_2] \geq \mathbb{E}^u[g_1] \mathbb{E}^u[g_2],$$

for *any* increasing functions $g_{1,2}$ with finite second moments. To complement the FKG inequality, we use *sprinkling*, i.e., we slightly change the intensity of random interlacements in order to decrease the error term; this approach was used in [17] and [18]. Then, in particular, in [13] it was proved that

$$(1.5) \quad \mathbb{E}^u[f_1 f_2] \leq \mathbb{E}^{(1+\varepsilon)u}[f_1] \mathbb{E}^{(1+\varepsilon)u}[f_2] + c_d (r + s)^d \exp(-c'_d \varepsilon^2 u s^{d-2});$$

with $f_1 : \{0, 1\}^{A_1} \rightarrow [0, 1]$ and $f_2 : \{0, 1\}^{A_2} \rightarrow [0, 1]$ both increasing functions in the interlacements set, $r = \min(\text{diam}(A_1), \text{diam}(A_2))$, and $s = \text{dist}(A_1, A_2)$. The same bound was also obtained for decreasing functions.

It is important to observe, however, that the decoupling in the above form may not always be useful for one's needs. Intuitively, one is tempted to understand inequalities like (1.3) as “what happens in one set does not influence a lot what happens in the other set”. Now, consider the following situation. Suppose that on top of the random interlacements we have some additional stochastic process (e.g., a random walk) that “explores” the interlacement set in some way. Assume that this process has already explored the interlacements in a given area, revealing a lot of information about it; think, for definiteness, that it simply revealed the interlacement set exactly. The probability of a particular configuration of the interlacement set is usually very small; so, (1.3) (even (1.5)!) will blow up when one divides by that probability, because of the error term. In fact, in the end of Section 2 we discuss a particular model of the random walk on the interlacement set, where our main results turn out to be useful.

This justifies the need for *conditional* decoupling, i.e., show that, given the configuration on some set, the law of the interlacement configuration on a distant set is still in some sense close to the unconditional law. This is what we are doing in this paper. To prove our results, the main method we use is a suitable modification (that allows dealing with conditional probabilities) of the *soft local time* method of [13]. We hope that this modification will be useful in other contexts, for instance, for dealing with the decoupling properties of the *loop measures* [3].

Another important observation is the following. There are strong connections between random interlacements and the Gaussian free field, see e.g. [19, 20]. In particular, there are decoupling inequalities similar to (1.3) and (1.5) for the Gaussian free field as well, see [12]. Notice, however, that the decoupling-with-sprinkling result for the Gaussian free field (Theorem 1.2 of [12]) is *already* conditional (the unconditional decoupling is obtained as a simple consequence, just by integration). On the other hand, note that the

error terms in the conditional decoupling in the main result of this paper (Theorem 2.1) are much worse than that of (1.5); related to this is the fact that in the conditional setting the minimal distance between sets that permits the result to work is much bigger. A comparison with the situation for the Gaussian free field suggests that, hopefully, there is still much room for improvement for the conditional decoupling for random interlacements.

2. DEFINITIONS, NOTATIONS AND RESULTS

In this section we will introduce the basic definitions, conventions and notation used in this paper. We will then be able to state our main result. We start by stating our convention regarding constants: $c, c', c_1, c_2, c_3, \dots$ are always defined as strictly positive constants depending only on the dimension d . Constants can also change value from line to line, unless when the text explicitly states to the contrary.

We let $\|\cdot\|$ and $\|\cdot\|_\infty$ denote the Euclidean and ℓ_∞ norms in \mathbb{Z}^d respectively. For $x, y \in \mathbb{Z}^d$, we also let $\text{dist}(x, y) \equiv \|x - y\|$. We say that two vertices $x, y \in \mathbb{Z}^d$ are neighbors when $\|x - y\| = 1$, this notion introduces the usual nearest-neighbor graph structure in \mathbb{Z}^d . For $x \in \mathbb{Z}^d$ and $r \in \mathbb{R}_+$, we define

$$B(x, r) := \{y \in \mathbb{Z}^d; \|y - x\| \leq r\},$$

the discrete ball in the Euclidean norm centered on x with radius r , and

$$B_\infty(x, r) := \{y \in \mathbb{Z}^d; \|y - x\|_\infty \leq r\},$$

the discrete ball in the ℓ_∞ -norm centered on x with radius r . Given a set $A \subseteq \mathbb{Z}^d$ we denote by

$$A^C := \{x \in \mathbb{Z}^d; x \notin A\}$$

its complement and by

$$\partial A := \{x \in A; \text{there exists } y \in A^C \text{ such that } \|x - y\| = 1\}$$

its (internal) boundary.

For any set Z and any two functions $f, g : Z \mapsto \mathbb{R}$, we write $f(z) \asymp g(z)$ to denote the fact that there exist two strictly positive constants, c_1 and c_2 , such that $c_1 f(z) \leq g(z) \leq c_2 f(z)$ for all $z \in Z$. When Z is equal to \mathbb{R} we say that $f(z) = o(g(z))$ when $\frac{f(z)}{g(z)}$ goes to 0 as $z \rightarrow \infty$.

Given $x \in \mathbb{Z}^d$, we let \mathbb{P}_x denote the probability measure associated with the simple random walk in \mathbb{Z}^d started at x . We will also let $(X_k, k \geq 0)$ denote the simple random walk process in \mathbb{Z}^d . Given a set $A \subset \mathbb{Z}^d$, we define the entrance time for the set A

$$H_A := \inf \{k \geq 0; X_k \in A\}.$$

We also let the hitting time for A be defined as

$$\tilde{H}_A := \inf \{k \geq 1; X_k \in A\}.$$

When A is finite we denote its harmonic measure by

$$e_A(x) = 1_{x \in A} \mathbb{P}_x[\tilde{H}_A = \infty] \text{ for } x \in \mathbb{Z}^d.$$

We are then able to define the capacity of the set A

$$\text{cap}(A) := \sum_{x \in A} e_A(x),$$

and the normalized harmonic measure

$$\bar{e}_A(x) := e_A(x) \text{cap}(A)^{-1}.$$

We now write down the definition of the Green's function for the simple random walk in \mathbb{Z}^d : for $x, y \in \mathbb{Z}^d$, we let

$$G(x, y) := \sum_{k \geq 0} \mathbb{P}_x[X_k = y].$$

Theorem 1.5.4 of [10] provides us with the following estimate on the Green's function:

$$(2.1) \quad G(x, y) \asymp \frac{1}{1 + \|x - y\|^{d-2}}.$$

Let us briefly discuss the definition of the measure associated with the random interacements process intersected with a given finite set $A \subset \mathbb{Z}^d$. Assume we have constructed a probability space where, for every $i \geq 1$, there exists a simple random walk process $(X_k^{(i)}, k \geq 0)$ with starting distribution given by $\bar{e}_A(\cdot)$, and such that $(X_k^{(i)}, k \geq 0)$ is independent from $(X_k^{(j)}, k \geq 0)$ for $i \neq j$. We also assume that in this space we can construct an independent Poisson process $(J_u)_{u \geq 0}$ on the positive real line with intensity $\text{cap}(A)$. The law of the random interacements process $(\mathcal{I}^u)_{u \geq 0}$ intersected with the set A can then be characterized by

$$(2.2) \quad (\mathcal{I}^u \cap A)_{u \geq 0} \stackrel{d}{=} \left(A \cap \bigcup_{i \leq J_u} \bigcup_{k \geq 0} X_k^{(i)} \right)_{u \geq 0},$$

as can be seen in [17], Proposition 1.3, or in the paragraph before (2.6) in [5]. This definition gives rise to compatible measures in the following sense: Given two finite sets $K_1 \subset K_2 \subset \mathbb{Z}^d$, we have that $((\mathcal{I}^u \cap K_2)_{u \geq 0}) \cap K_1$ has the same law as $(\mathcal{I}^u \cap K_1)_{u \geq 0}$.

To state our main result, we need more definitions. Let $r > 0$ be sufficiently big, and

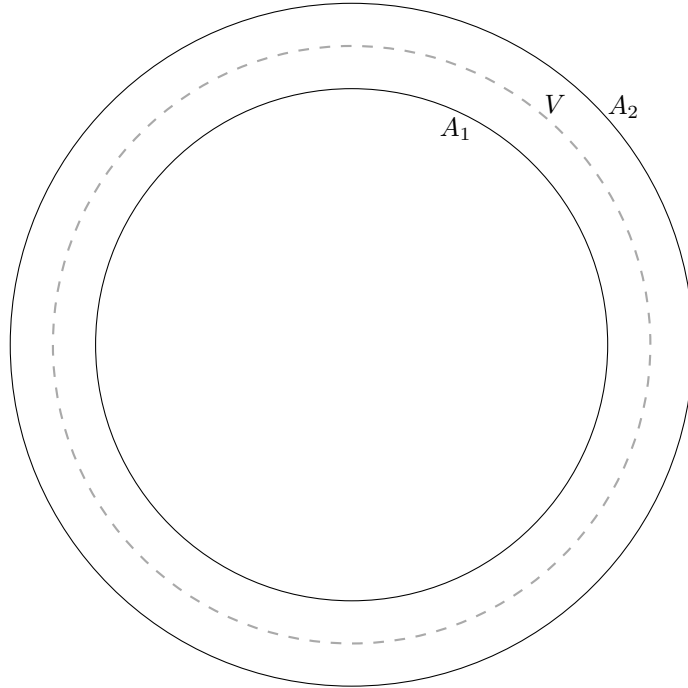


FIGURE 1. Definition of the sets A_1° , A_2° and V° .

let $s := s(r) > 0$, with $s = o(r)$. We define $A_1^\circ := A_1^\circ(r)$ to be the discrete ball of radius r , that is

$$A_1^\circ := \{x_1 \in \mathbb{Z}^d; \text{dist}(x_1, 0) < r\}.$$

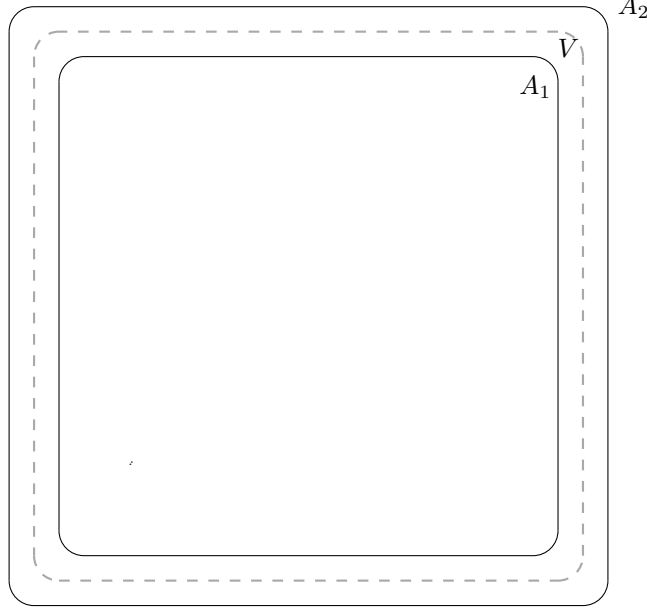


FIGURE 2. Definition of the sets A_1^\square , A_2^\square and V^\square .

We also define $A_1^\square := A_1^\square(r, s)$ to be a d -dimensional discrete ‘hypercube’ with edge length r and a smoothed frontier such that for every point $x_1 \in \partial A_1$ there exists a discrete Euclidean ball B_{x_1} of radius s contained in A_1 such that $B_{x_1} \cap A_1^C = x_1$. More precisely, we let \mathfrak{H}_{r-s} be a discrete d -dimensional hypercube with edge length $r - s$ contained in \mathbb{Z}^d and define

$$A_1^\square := \{x_1 \in \mathbb{Z}^d; \text{dist}(x_1, \mathfrak{H}_{r-s}) \leq s\}.$$

We refer the reader to [13], Section 8, to see that A_1^\square possesses the desired properties. Note that, since $s = o(r)$, the diameter of A_1^\square is of order r .

We then define $A_2^\square := A_2^\square(r, s)$ to be the set of points that are at least at distance $2s$ from A_1^\square :

$$A_2^\square := \{x_1 \in \mathbb{Z}^d; \text{dist}(x_1, x_2) > 2s \text{ for every } x_2 \in A_1^\square\}.$$

We finally define $V^\square := V^\square(r, s)$ to be the boundary set

$$V^\square := \partial\{x_1 \in \mathbb{Z}^d, \text{dist}(x_1, x_2) \leq s \text{ for some } x_2 \in A_1^\square\},$$

separating A_1^\square from A_2^\square . We analogously define $A_2^\square(r, s)$ and $V^\square(r, s)$. It will also be useful to define the d -dimensional hypercube \mathfrak{H}_{r+2s} of edge length $r + 2s$ concentric with \mathfrak{H}_{r-s} , which will essentially be the unsmoothed version of $(A_2^\square)^C$.

When there is no risk of confusion, or when the arguments presented work for both balls and smoothed hypercubes (which will be often so), we will omit the super-indexes \circ, \square .

Since $s = o(r)$, we have

$$\text{cap}(V) = \text{cap}(A_2)(1 + o(1)) = \text{cap}(A_1)(1 + o(1)),$$

and also, by Proposition 2.2.1 and equation (2.16) of [10],

$$(2.3) \quad \text{cap}(V) \asymp r^{d-2}.$$

We will now state our main result. Heuristically, it says the following: Let s be bounded from below by a polynomial of r with a explicit given coefficient (strictly smaller than 1, depending only on the dimension d and whether A_1 is a ball or a smoothed hypercube). Let A_3 be a subset of A_2 with finite boundary, that is, A_3 is either finite or has finite

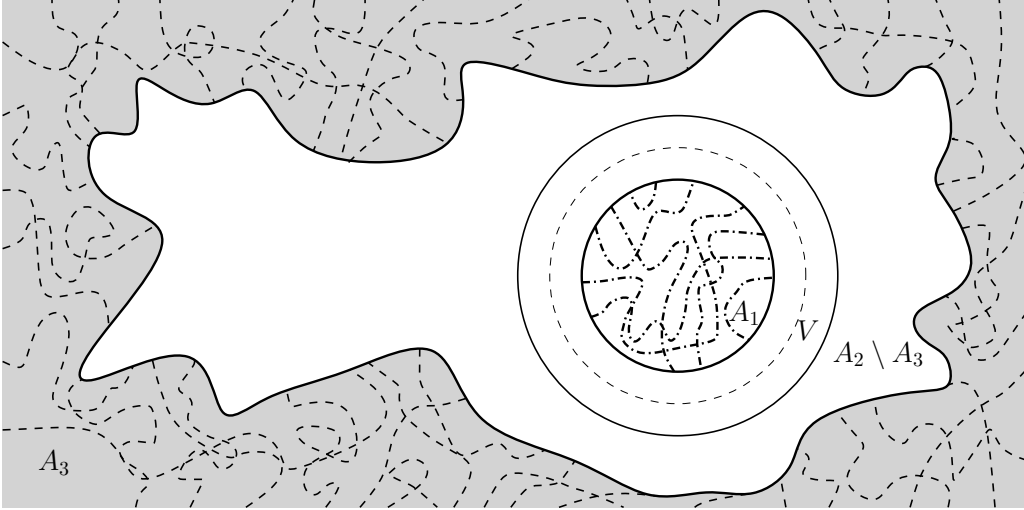


FIGURE 3. Our main result says that if the interlacements configuration in a set $A_3 \subseteq A_2$ is not too weird, that is, it does not belong to a set with stretched exponentially small probability (in s , as $s \rightarrow \infty$), then with high probability (1 minus stretched exponential in s) the distribution of the interlacements set intersected with A_1 conditioned on the state of $\mathcal{I}^u \cap A_3$ can be well approximated by the unconditional distribution.

complement. If we pay a stretched exponentially small price (in s) to guarantee that the interlacements configuration of $\mathcal{I}^u \cap A_3$ is not too weird, then the distribution of $\mathcal{I}^u \cap A_1$ conditioned on this configuration is well approximated by the unconditional distribution, with high probability (1 minus a stretched exponential function of s).

Theorem 2.1. *Let the real numbers $b_{A_1^\circ}, b_{A_1^\square}$ be such that*

$$(2.4) \quad 1 \leq b_{A_1^\circ} < \frac{2d-2}{d},$$

$$(2.5) \quad 1 \leq b_{A_1^\square} < \frac{4d-4}{3d-2}.$$

Then, define

$$(2.6) \quad a_{A_1^\circ} = 2d-2 - db_{A_1^\circ} > 0,$$

$$(2.7) \quad a_{A_1^\square} = 4d-4 - 3db_{A_1^\square} + 2b_{A_1^\square} > 0.$$

From now on we will again omit the indexes \circ, \square . Recall that r is of the same order as the diameter of A_1 , and that s has the same order as the distance between A_1 and A_2 . Assume $r \asymp s^{b_{A_1}}$, let s be sufficiently big. Let $\varepsilon > 0$ be smaller than $1/4$. Let A_3 be a subset of A_2 such that $|\partial A_3| < \infty$. Define $\mathcal{I}_{A_j}^u := \mathcal{I}^u \cap A_j$, for $j = 1, 2, 3$.

Then there are positive constants c, c' depending only on the dimension d , and a measurable (according to the random interlacements σ -field) set $\mathcal{G} \in \{0, 1\}^{A_3}$ such that

$$\mathbb{P}^u[\mathcal{I}_{A_3}^u \in \mathcal{G}] \geq 1 - \exp\left(-\frac{c'}{2}\varepsilon^2 u s^{a_{A_1}}\right),$$

and for any increasing function f on the interlacements set intersected with A_1 , with $\sup |f| < M$, we have

$$\begin{aligned}
(\mathbb{E}f(\mathcal{I}_{A_1}^{u(1-\varepsilon)}) - cM \exp(-c'\varepsilon^2 u s^{a_{A_1}})) 1_{\mathcal{I}_{A_3}^u \in \mathcal{G}} &\leq \mathbb{E}(f(\mathcal{I}_{A_1}^u) | \mathcal{I}_{A_3}^u) 1_{\mathcal{I}_{A_3}^u \in \mathcal{G}} \\
(2.8) \qquad \qquad \qquad &\leq (\mathbb{E}f(\mathcal{I}_{A_1}^{u(1+\varepsilon)}) + cM \exp(-c'\varepsilon^2 u s^{a_{A_1}})) 1_{\mathcal{I}_{A_3}^u \in \mathcal{G}}.
\end{aligned}$$

We also obtain a result analogous to Theorem 2.1, but this time we allow the sprinkling factor to be arbitrarily big. This decreases the “precision” (in the result below, $\mathbb{E}f(\mathcal{I}_{A_1}^{u+u'})$ can be very different from $\mathbb{E}f(\mathcal{I}_{A_1}^u)$), but, in compensation, the size of the complement of the “good” set as well as the “error term” become smaller.

Theorem 2.2. *Let $u' > u > 0$. We use the same definitions as Theorem 2.1. There are positive constants c, c' depending only on the dimension d , and a measurable (according to the random interacements σ -field) set $\mathcal{G}_{u'} \in \{0, 1\}^{A_3}$ such that*

$$\mathbb{P}^u[\mathcal{I}_{A_3}^u \in \mathcal{G}_{u'}] \geq 1 - \exp(-c'u's^{a_{A_1}}),$$

and for any increasing function f on the interacements set intersected with A_1 , with $\sup|f| < M$, we have

$$(2.9) \qquad \mathbb{E}(f(\mathcal{I}_{A_1}^u) | \mathcal{I}_{A_3}^u) 1_{\mathcal{I}_{A_3}^u \in \mathcal{G}_{u'}} \leq (\mathbb{E}f(\mathcal{I}_{A_1}^{u+u'}) + cM \exp(-c'u's^{a_{A_1}})) 1_{\mathcal{I}_{A_3}^u \in \mathcal{G}_{u'}}.$$

Remark 2.3. We have to explain why we need to consider $A_3 \subset A_2$. Indeed, at first sight it seems that conditioning on a configuration on A_3 does not add generality to our results, since any fixed configuration on A_3 corresponds to a set of configurations on A_2 . However, the problem with always setting $A_3 = A_2$ is the following: the “exceptional set” \mathcal{G}^c will then be supported on the whole A_2 , and this can be inconvenient for applications. For example, assume that we successively apply the conditional decoupling results to a process (such as the one of Section 2.1) that “explores” the interlacement environment. If that process has explored only a finite chunk of A_2 , we would not be able to say if the configuration is “good” (i.e., belongs to \mathcal{G}) by only observing that finite chunk. This would force us to condition on the (configuration on the) whole A_2 , which would mean that a subsequent application of a conditional decoupling may be difficult, since we already “revealed” some information about the configuration on a set which is “too big” (i.e., when we apply the decoupling result for the next time, the “new” A_1 may be inside the “previous” A_2)

Remark 2.4. In the course of the proof of the above theorems we actually prove a stronger result: the same conditional decoupling inequality holds true if we replace the sets $\mathcal{I}_{A_1}^u \subset A_1$ and $\mathcal{I}_{A_3}^u \subset A_3$ by sets of *random walk excursions* in A_1 and A_3 (we also have to replace the function f by an increasing function on the set of excursions). That is, the conditional decoupling continues to work when we replace the ranges of the excursions (which constitute the random interacements set) by the actual excursions themselves. We chose to state the results in the above manner for the sake of clarity and brevity. Note that this remark also applies to the decoupling obtained by Popov and Teixeira in [13].

Remark 2.5. The above theorems can be proved in the same way if we replace the smoothed hypercube A_1^\square by a smoothed version of a box $[0, a_1] \times \dots \times [0, a_d]$, with $c^{-1}r < a_i < cr$ for all $i = 1, \dots, d$, and some constant $c > 1$, and then replace the sets A_2^\square and V^\square accordingly. We chose to prove the theorems for A_1^\square only to simplify the notation. We also note that we prove the theorem for both balls and boxes because the error term obtained in the decoupling for balls is much smaller than the error obtained in the decoupling for boxes, but at the same time the decoupling between boxes tends to be more useful because boxes cover the space in a much more efficient manner.

Remark 2.6. For $d = 3$, the only way to obtain an exponentially small (instead of a *stretched exponentially* small) error term in equations (2.8) and (2.9) is to allow the distance $\sim s$ between the sets A_1 and A_2 to be of the same order of the minimal diameter $\sim r$.

Here is an overview of the paper. In Subsection 2.1, we discuss an application of some of our results. In Section 3 we recall the soft local times technique. In Section 4 we show how we simulate the interacements set $\mathcal{I}_{A_1}^u$ conditioned on the information given by $\mathcal{I}_{A_2}^u$ using a suitable version of the soft local times method. Finally, in Section 5, we prove the main theorem using a large deviations estimate for the soft local times associated with $\mathcal{I}_{A_1}^u$. The Appendix is then used to collect and derive the technical estimates we need.

2.1. An application: biased random walk on the interlacement set. Let G be some (possibly random) subset of \mathbb{Z}^d , $d \geq 2$. Fix a parameter $\beta > 0$, which accounts for the bias; also, fix some non-zero vector $\ell \in \mathbb{Z}^d$. Let us define the *conductances* on the edges of \mathbb{Z}^d in the following way:

$$\mathcal{C}(x, y) = \begin{cases} e^{\beta(x+y) \cdot \ell}, & \text{if } x, y \text{ are neighbors and belong to } G, \\ 0, & \text{otherwise,} \end{cases}$$

and we call the collection of all conductances $\omega = \{\mathcal{C}(x, y), x, y \in \mathbb{Z}^d\}$ the random environment. Consider a random walk $(X_n, n \geq 0)$ in this environment of conductances; i.e., its transition probabilities are given by

$$P^\omega[X_{n+1} = y \mid X_n = x] = \frac{\mathcal{C}(x, y)}{\sum_z \mathcal{C}(x, z)}$$

(the superscript in P^ω indicates that we are dealing with the “quenched” probabilities, i.e., when the underlying random graph / conductancies are already fixed).

There have been significant interest towards this model in recent years, mainly in the case when G is the infinite cluster of supercritical Bernoulli percolation model, see e.g. [1, 16, 7]. In particular, one remarkable fact is the following: the walk is ballistic (transient and with positive speed) in the direction of the drift if $\beta > 0$ is small enough; however, it moves only sublinearly fast (its displacement is only of order t^a by time t with $a \in (0, 1)$), as proved in [8] for large values of β .

In the work [9] the case $G = \mathcal{I}^u$ was considered. It turned out that in dimension $d = 3$, for *any* value of $\beta > 0$, although still transient in the direction of the drift, the walk is not only sub-ballistic, but has also sub-polynomial speed, in the sense that its distance to the origin grows slower than t^ε for any $\varepsilon > 0$. This is also in contrast with the result that the walk on \mathcal{I}^u without any drift is diffusive (so, loosely speaking, its “speed” is \sqrt{t}), as shown in [14].

We will not describe all the details of [9] here, but the main idea is the following. As in the case of the biased walk on the infinite percolation cluster, to prove zero speed one needs to show that the walk frequently gets caught in traps. These traps are “dead ends” of the environment looking in the direction of the bias, see Figure 4. When the walk enters such a trap, the bias prevents it from going out, so there is a good chance that the walk will spend quite a lot of time there, and this effectively leads to zero speed. Now, the crucial fact is that, specifically in three dimensions, it is much cheaper to have a trap in the interlacement set than in the (Bernoulli) percolation cluster. Indeed, it is possible to show that the capacity of the dotted set on Figure 4 is of order $\frac{\ln t}{\ln \ln t}$ for any fixed $\alpha < 1$. The formula (1.1) then shows that having a trap as above has only a subpolynomial (in t)

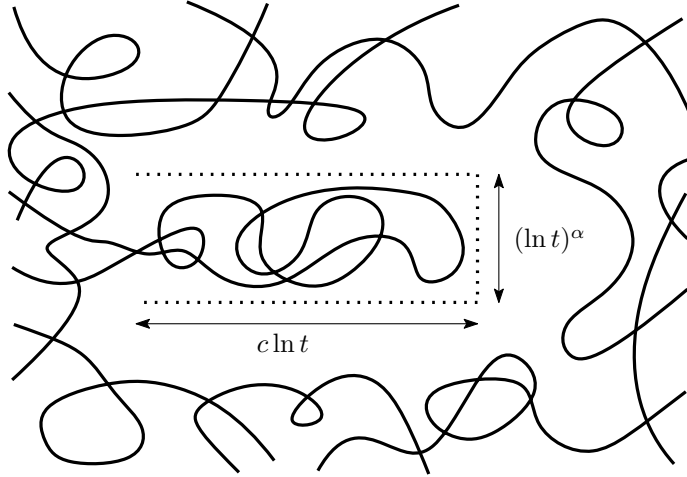


FIGURE 4. A trap for the random walk on the interlacement set (on this picture, the bias is directed along the first coordinate vector). Only the interlacements are shown; the trajectory of the RWRE X is not present on the picture.

cost; also, it turns out that “forcing” a trajectory to create a “dead end” as shown on the picture is not too costly as well.

So, when the walk advances in the direction of the bias, from time to time it will encounter a trap and be trapped. However, to make such an argument rigorous, one has to face the following difficulty. When the walk already explored some parts of the environment and then came to an unexplored area, we can no longer use (1.1) to estimate the probability that there is a trap in front of it, due to the lack of independence. It is here that the conditional decoupling enters the scene: it is possible to use the main results of this paper to show that probability of having a trap in front of the particle (when it comes to an unexplored area) is not very small. As mentioned above, the detailed argument can be found in [9].

3. SOFT LOCAL TIMES

In the present section we describe the technique introduced in [13], the so called Soft Local Times method. This method essentially allows us to simulate any number of random variables taking values in a state space Σ using a realization of a Poisson point process in $\Sigma \times \mathbb{R}_+$.

Let Σ be a locally compact Polish metric space, and let $\mathcal{B}(\Sigma)$ be its Borel σ -algebra. Let μ be a Radon measure over $\mathcal{B}(\Sigma)$, so that every compact set has finite μ -measure.

Such measure space $(\Sigma, \mathcal{B}(\Sigma), \mu)$ is the usual setup for the construction of a Poisson point process on Σ . We consider the space of Radon point measures in $\Sigma \times \mathbb{R}_+$

$$(3.1) \quad L = \left\{ \eta = \sum_{\lambda \in \Lambda} \delta_{(z_\lambda, v_\lambda)}; z_\lambda \in \Sigma, v_\lambda \in \mathbb{R}_+ \text{ and } \eta(K) < \infty \text{ for all compact } K \right\},$$

endowed with the σ -algebra generated by the evaluation maps

$$\eta \mapsto \eta(D), \quad D \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\Sigma).$$

We are then able to construct a Poisson point process η in the space $(L, \mathcal{D}, \mathbb{Q})$ with intensity measure given by $\mu \otimes dv$, where dv is the Lebesgue measure on \mathbb{R}_+ , see [15], Proposition 3.6 on p.130.

The next proposition, originally seen in [13], is at the core of the soft local times argument.

Proposition 3.1. *Let $g : \Sigma \rightarrow \mathbb{R}_+$ be a measurable function with $\int g(z)\mu(dz) = 1$. For $\eta = \sum_{\lambda \in \Lambda} \delta_{(z_\lambda, v_\lambda)} \in L$, we define*

$$(3.2) \quad \xi = \inf\{t \geq 0; \text{ there exists } \lambda \in \Lambda \text{ such that } tg(z_\lambda) \geq v_\lambda\}.$$

Then under the law \mathbb{Q} of the Poisson point process η ,

- (i) *there exists a.s. a unique $\hat{\lambda} \in \Lambda$ such that $\xi g(z_{\hat{\lambda}}) = v_{\hat{\lambda}}$,*
- (ii) *$(z_{\hat{\lambda}}, \xi)$ is distributed as $g(z)\mu(dz) \otimes \text{Exp}(1)$,*
- (iii) *$\eta' := \sum_{\lambda \neq \hat{\lambda}} \delta_{(z_\lambda, v_\lambda - \xi g(z_\lambda))}$ has the same law as η and is independent of $(\xi, \hat{\lambda})$.*

The proof is remarkably simple, mainly relying on the independence of a Poisson process in disjoint sets, and can be seen in the original paper.

With the above proposition we are able to simulate as many random variables as we want:

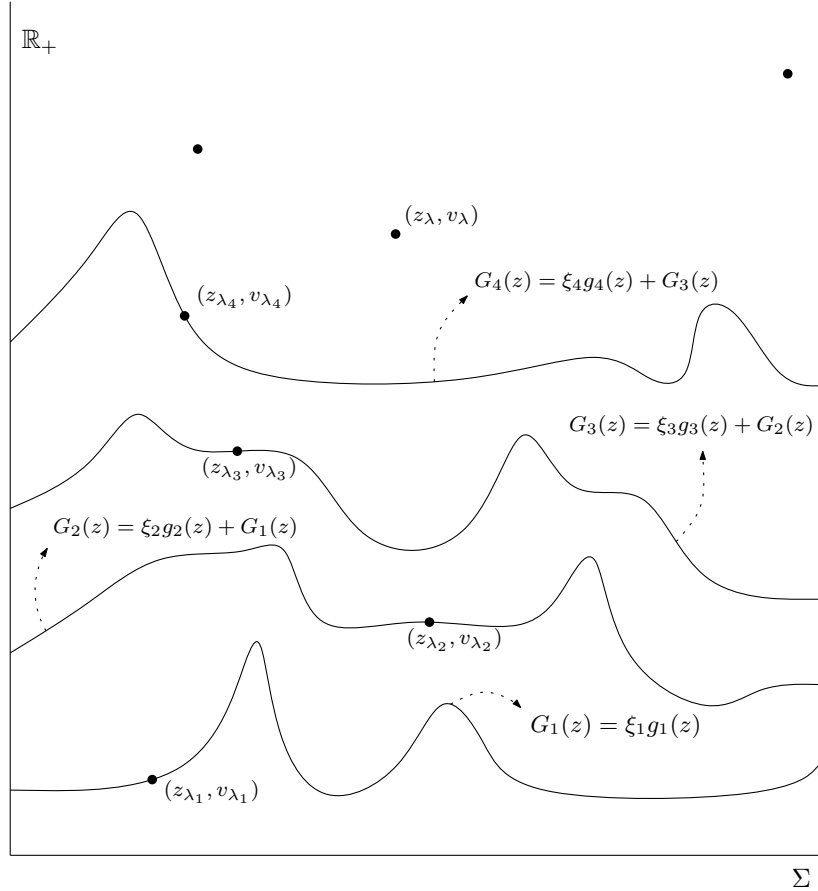


FIGURE 5. An example showing the definition below. Under mild conditions we are able to use Proposition 3.1 to simulate a sequence of random variables over Σ .

Let X_1, X_2, \dots, X_n be random variables on Σ such that X_1 's distribution is absolutely continuous with respect to μ and, for all $i = 2, \dots, n$ the probability measure generated

by X_i , conditioned on the values taken by X_1, \dots, X_{i-1} , is absolutely continuous with respect to μ . Using the process η constructed above, we define

$$(3.3) \quad \begin{aligned} g_1 &: \Sigma \mapsto \mathbb{R}_+, \text{ the density function of } X_1 \text{ with respect to } \mu, \\ \xi_1 &:= \inf \{t \geq 0; \text{ there exists } \lambda \in \Lambda \text{ such that } tg_1(z_\lambda) \geq v_\lambda\}, \\ G_1(z) &:= \xi_1 g_1(z), \text{ for } z \in \Sigma, \\ (z_{\lambda_1}, v_{\lambda_1}) &, \text{ the unique pair in } \{(z_\lambda, v_\lambda)\}_{\lambda \in \Lambda} \text{ with } G_1(z_{\lambda_1}) = v_{\lambda_1}. \end{aligned}$$

We now define $g_2 : \Sigma \mapsto \mathbb{R}_+$ to be the density of X_2 conditioned on the event $\{X_1 = z_{\lambda_1}\}$. Using the fact that $\eta_1 := \sum_{\lambda \neq \lambda_1} \delta_{(z_\lambda, v_\lambda - \xi_1 g_1(z_\lambda))}$ has the same law as η and is independent from (ξ_1, λ_1) we define

$$(3.4) \quad \begin{aligned} \xi_2 &:= \inf \{t \geq 0; \text{ there exists } \lambda \in \Lambda \text{ such that } tg_2(z_\lambda) + G_1(z_\lambda) \geq v_\lambda\}, \\ G_2(z) &:= \xi_2 g_2(z) + G_1(z), \text{ for } z \in \Sigma, \\ (z_{\lambda_2}, v_{\lambda_2}) &, \text{ the unique pair in } \{(z_\lambda, v_\lambda)\}_{\lambda \in \Lambda} \text{ with } G_2(z_{\lambda_2}) = v_{\lambda_2}. \end{aligned}$$

Then, recursively, for $1 \leq k \leq n$ we define $g_k : \Sigma \mapsto \mathbb{R}_+$ to be the density function of X_k conditioned on the event $\{X_1 = z_{\lambda_1}, \dots, X_{k-1} = z_{\lambda_{k-1}}\}$,

$$(3.5) \quad \begin{aligned} \xi_k &:= \inf \{t \geq 0; \text{ there exists } \lambda \in \Lambda \text{ such that } tg_k(z_\lambda) + G_{k-1}(z_\lambda) \geq v_\lambda\}, \\ G_k(z) &:= \xi_k g_k(z) + G_{k-1}(z), \text{ for } z \in \Sigma, \\ (z_{\lambda_k}, v_{\lambda_k}) &, \text{ the unique pair in } \{(z_\lambda, v_\lambda)\}_{\lambda \in \Lambda} \text{ with } G_k(z_{\lambda_k}) = v_{\lambda_k}. \end{aligned}$$

We refer to Figure 5. Using Proposition 3.1 together with the above construction, we are able to state the following proposition:

Proposition 3.2. *The vector $(z_{\lambda_1}, \dots, z_{\lambda_n})$ has the same law as (X_1, \dots, X_n) .*

We call the function $G_n(z)$ the soft local time of the vector (X_1, \dots, X_n) up to time n with respect to the measure μ , or more usually simply the soft local time. If T is a stopping time with respect to the canonical filtration generated by the variables X_i , it is simple to define $G_T(z)$, the soft local time up to time T .

Note that by controlling the value of the soft local times function we will automatically control the values our random variables take, as the next corollary summarizes:

Corollary 3.3. *For any measurable function $h : \Sigma \rightarrow \mathbb{R}_+$ we have, using the same notation as above,*

$$(3.6) \quad \mathbb{Q}[\{z_1, \dots, z_T\} \subseteq \{z_\lambda; v_\lambda \leq h(z_\lambda)\}] \geq \mathbb{Q}[G_T(z) \leq h(z), \text{ for } \mu\text{-a.e. } z \in \Sigma],$$

for any finite stopping time $T \geq 1$.

4. SIMULATING EXCURSIONS

In this section we will show a way of simulating the intersection of the random interlacements set with a given subset of \mathbb{Z}^d in such a way as to make explicit the dependence each random walk excursion has with its entrance and exit points on the subset. We refer the reader to Figure 6 for a brief overview of the arguments used in this section.

It is clear from (2.2) the fact that in order to simulate the random interlacements set at level u in a bounded subset K of \mathbb{Z}^d we need only to pick a $N_K^u \stackrel{d}{=} \text{Poisson}(u \text{cap}(K))$ number of points in ∂K , each point chosen according to the measure $\bar{e}_K(\cdot)$, and from each point start a simple random walk.

We intend to study $\mathcal{I}_{A_1}^u = \mathcal{I}^u \cap A_1$, showing that this set is not much influenced by the random interlacements set intersected with A_2 , $\mathcal{I}_{A_2}^u = \mathcal{I}^u \cap A_2$. We will later clarify what

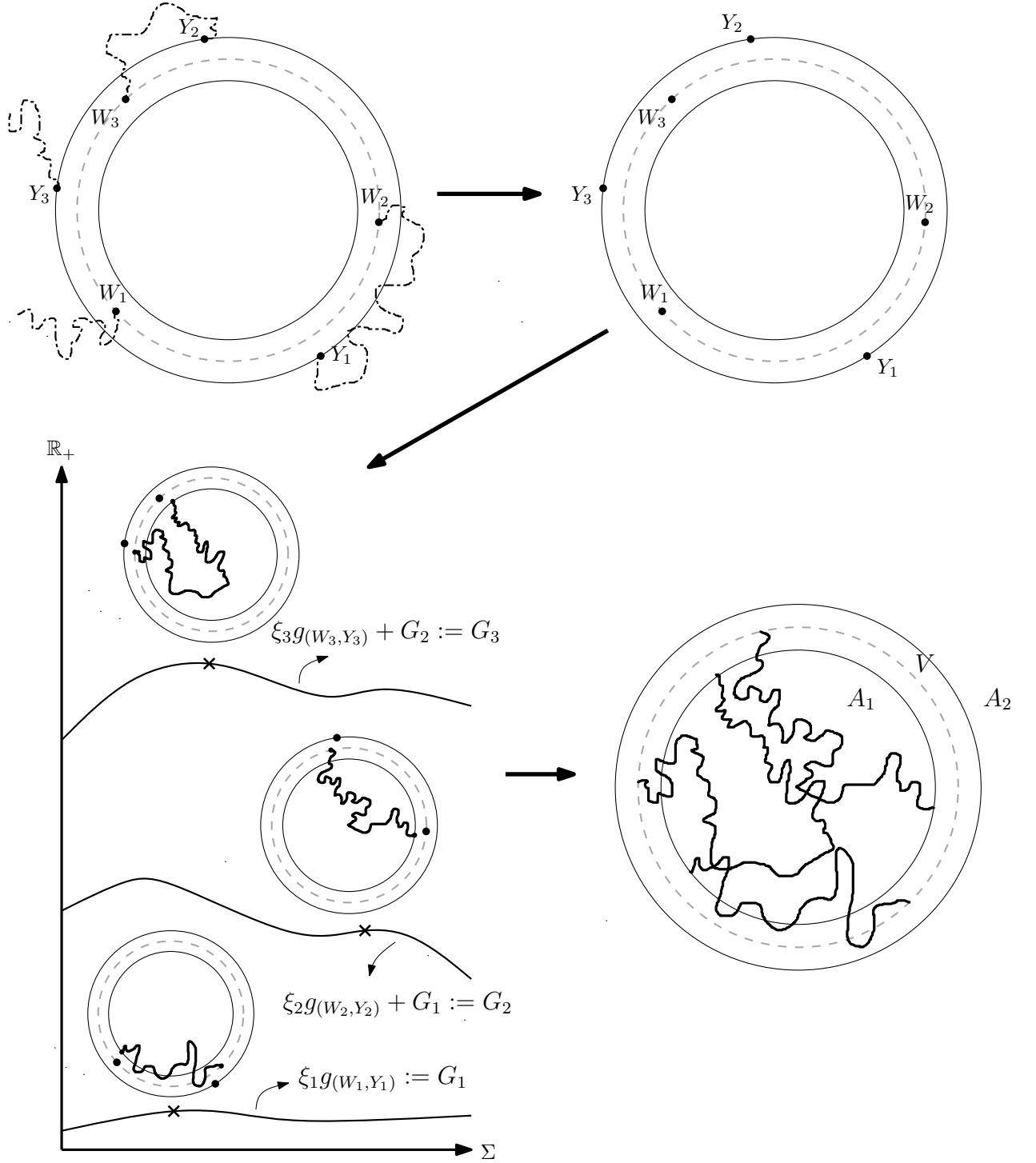


FIGURE 6. The figure shows how we will use the soft local times technique to simulate the range of a simple random walk trajectory intersected with A_1 . We first simulate a process of pairs of points $((W_k, Y_k), k \geq 0)$ denoting the entrance at V and exit at ∂A_2 of a simple random walk trajectory that starts at V . We then use the soft local times method to simulate the pieces of trajectory that lie between each of the pairs (W_k, Y_k) .

we mean by “influence”. For now, we observe that the only “information” $\mathcal{I}_{A_1}^u$ receives

from $\mathcal{I}_{A_2}^u$ is the location of the entrance and exit points of the excursions on ∂A_2 of the random walks that constitute $\mathcal{I}_{A_2}^u$.

Let us begin the work towards our result. We first generate the points of entrance at V and exit from A_2^C of each excursion on V of a random walk trajectory. These points will be the clothesline onto which we will hang the pieces of trajectory that meet A_1 , we will do so using the soft local times method.

Let us define the successive return and departure times between V and A_2 . Given a trajectory that starts at V , we define

$$(4.1) \quad \begin{aligned} D_0 &= 0, & R_1 &= H_{\partial A_2}, \\ D_1 &= H_V \circ \theta_{R_1} + R_1, & R_2 &= H_{\partial A_2} \circ \theta_{D_1} + D_1, \\ D_2 &= H_V \circ \theta_{R_2} + R_2 & & \text{and so on.} \end{aligned}$$

We also define the random time

$$(4.2) \quad T_\Delta = \inf\{k \geq 1; R_k = \infty\},$$

which is almost surely finite, as the walk is transient.

Let $(X_n, n \geq 0)$ be the simple random walk with initial distribution given by $\bar{e}_V(\cdot)$. Let Δ be an artificial cemetery state. We construct a random sequence of elements of $(V \times \partial A_2) \cup \{\Delta\}$ in the following way: Conditioned on the event $\{T_\Delta = m\}$, we let

$$\begin{aligned} &((W_1, Y_1), \dots, (W_{m-1}, Y_{m-1}), (W_m, Y_m), (W_{m+1}, Y_{m+1}), \dots) \\ &= ((X_{D_0}, X_{R_1}), \dots, (X_{D_{m-2}}, X_{R_{m-1}}), \Delta, \Delta, \dots). \end{aligned}$$

It is then elementary to prove that the process $((W_k, Y_k))_{k \geq 1}$ inherits the Markov property from the simple random walk. We call $((W_k, Y_k))_{k \geq 1}$ the clothesline process started at W_1 . When there is no risk of confusion we will also denote by \mathbb{P}_{w_0} the probability measure associated with the clothesline process started at a given point $w_0 \in V$.

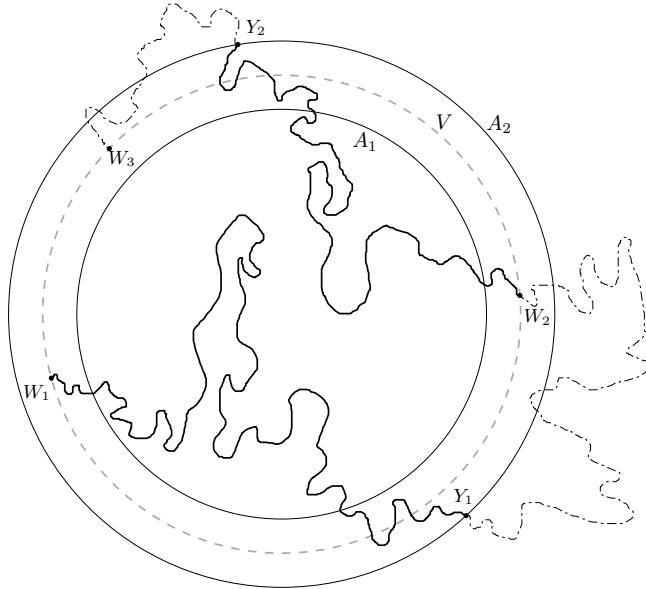


FIGURE 7. An example of the process $((W_k, Y_k))_{k \geq 1}$.

Let us now use the soft local times method to generate the trajectories inside A_1 , given the entrance and exit points $((W_k, Y_k))_{k \geq 1}$. We first define the underlying space Σ where

our pieces of trajectories will live. We let \mathcal{K} be the set of nearest-neighbor paths in A_2^C with one endpoint in ∂A_1 and the other in V ,

$$(4.3) \quad \mathcal{K} := \{(x_0, x_1, \dots, x_n); n \in \mathbb{N}, x_i \in A_2^C \text{ for } 1 \leq i \leq n, x_0 \in \partial A_1, x_n \in V\}.$$

We introduce yet another artificial state Θ for reasons that will be made clear in a few

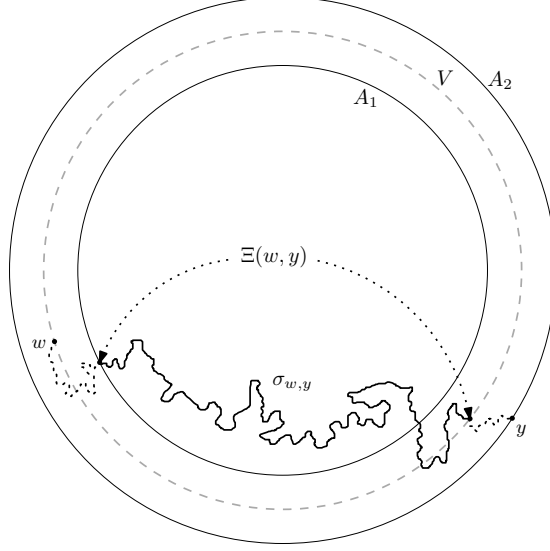


FIGURE 8. The definition of $\sigma(w, y)$ and $\Xi(w, y)$.

moments. We let $\Sigma := \mathcal{K} \cup \{\Theta\}$ and let μ be a measure on Σ defined in the following way: given $A \subseteq \Sigma$,

$$(4.4) \quad \mu(A) := \sum_{(x_0, \dots, x_n) \in A} \mathbb{P}_{(x_0, x_n)}[X_0 = x_0, \dots, X_n = x_n] + 1_{\{\Theta \in A\}},$$

where $\mathbb{P}_{(x_0, x_n)}$ is the simple random walk measure conditioned on the event where x_0 is the walk's initial point and x_n is its last point on V before reaching ∂A_2 . Notice that $\mu(\{\Theta\}) = 1$.

Given $(w, y) \in V \times \partial A_2$ we let $\mathbb{P}_{w, y}$ be the measure associated with simple random walk starting at w conditioned on the event where y is the first point the walk hits in ∂A_2 , that is:

$$(4.5) \quad \mathbb{P}_{w, y}[\cdot] := \mathbb{P}_w[\cdot \mid X_{H_{\partial A_2}} = y]$$

We want to randomly select (according to the conditional simple random walk measure above) a piece of trajectory in A_1 given a starting point in V and an ending point in ∂A_2 . Given $w \in V$ and $y \in \partial A_2$ we define the random element $\sigma_{w, y} \in \Sigma$ in the following way:

- Let $\mathcal{B}_{w, y}$ be a Bernoulli random variable with parameter $\mathbb{P}_{w, y}[H_{\partial A_1} < H_{\partial A_2}]$.
- If $\mathcal{B}_{w, y} = 0$ we let $\sigma_{w, y} \equiv \Theta$.
- If $\mathcal{B}_{w, y} = 1$ we let, for $\mathfrak{A} \subseteq \mathcal{K}$:

$$(4.6) \quad \mathbb{P}[\sigma_{w, y} \in \mathfrak{A}] = \sum_{(a_0, \dots, a_n) \in \mathfrak{A}} \mathbb{P}_{w, y} \left[\begin{array}{l} X_{H_{A_1}} = a_0, X_{H_{A_1}+1} = a_1, \dots, X_{H_{A_1}+n} = a_n, \\ X_k \notin A_1 \text{ for every } k = H_{A_1} + n + 1, \dots, H_{A_2} \end{array} \right].$$

In other words, the random element $\sigma_{w, y} \in \Sigma$ will either be Θ , on the event where a random walk starting at w and exiting at y fails to reach A_1 , or a simple random walk trajectory $(x_0^{w, y}, x_1^{w, y}, \dots, x_{k(w, y)}^{w, y}) \in \mathcal{K}$ distributed so that $x_0^{w, y}$ is the first point in A_1

after the start at w and $x_{k(w,y)}^{w,y}$ is the last point in V before reaching $y \in \partial A_2$. We then define $g_{(w,y)} : \Sigma \mapsto \mathbb{R}_+$ to be the μ -density of $\sigma_{w,y}$. We refer to Figure 8.

Given $z = (x_0, \dots, x_n) \in \mathcal{K}$ we denote by $\Xi(z)$ the pair (x_0, x_n) , the path's starting and ending points. We also let $\Xi(\Theta) = \Theta$ so that $\Xi(z)$ is defined for all $z \in \Sigma$. For $(w, y) \in V \times \partial A_2$ we define $\Xi(w, y)$ to be the random element $\Xi(\sigma_{w,y})$.

Let us calculate $g_{(w,y)}$ using the above notation. For $\mathfrak{A} \subseteq \Sigma$ we want to express the probability $\mathbb{P}[\sigma_{w,y} \in \mathfrak{A}]$ as a μ -integral over \mathfrak{A} .

$$\begin{aligned}
(4.7) \quad \mathbb{P}[\sigma_{w,y} \in \mathfrak{A}] &= \sum_{a \in \mathfrak{A}} \mathbb{P}[\sigma_{w,y} = a] \\
&= 1_{\{\Theta \in \mathfrak{A}\}} \mathbb{P}_{w,y}[\Xi(w, y) = \Theta] \\
&\quad + \sum_{\substack{a \in \mathfrak{A} \\ a \neq \Theta}} \mathbb{P}_{w,y}[\Xi(w, y) = \Xi(a)] \mathbb{P}_{w,y}[a \mid \Xi(w, y) = \Xi(a)] \\
&= 1_{\{\Theta \in \mathfrak{A}\}} \mathbb{P}_{w,y}[\Xi(w, y) = \Theta] + \sum_{\substack{a \in \mathfrak{A} \\ a \neq \Theta}} \mathbb{P}_{w,y}[\Xi(w, y) = \Xi(a)] \mathbb{P}_{\Xi(a)}[a] \\
&= \sum_{a \in \mathfrak{A}} \mathbb{P}_{w,y}[\Xi(w, y) = \Xi(a)] \mu(a) \\
&= \int_{\mathfrak{A}} \mathbb{P}_{w,y}[\Xi(w, y) = \Xi(z)] \mu(dz),
\end{aligned}$$

so that $g_{(w,y)}(z) = \mathbb{P}_{w,y}[\Xi(w, y) = \Xi(z)]$. Notice that the function $g_{(w,y)}(z)$ only depends on the pair $\Xi(z)$, the path's initial and ending points.

Let $(L, \mathcal{D}, \mathbb{Q})$ be the measure space of the Poisson point process on $\Sigma \times \mathbb{R}_+$ with intensity measure $\mu \otimes dv$, where dv is the Lebesgue measure on \mathbb{R}_+ . A weighted sum of functions $g_{(\cdot, \cdot)}$ indexed by clothesline processes $((W_k, Y_k))_{k \geq 1}$ will be the soft local time used to simulate the pieces of trajectory we need. This way we will be able to simulate the intersection of a simple random walk trajectory with A_1 . As we have seen in the random interacements process's definition, to simulate the interacements set inside V we need a number $N_V^u \stackrel{d}{=} \text{Poisson}(u \text{cap}(V))$ of independent random walks. We will need the same number of independent clothesline processes. For such task we will need a much bigger probability space, easily definable as a product between the Poisson point process space and an infinite product of independent simple random walk spaces starting on V . We call this bigger space the global probability space, and denote by \mathcal{P} its probability measure, which we will call the 'global probability'.

Given a clothesline process $((W_k, Y_k))_{k \geq 1}$, we define the trajectory's soft local time:

$$(4.8) \quad G(z) = \sum_{k=1}^{T_\Delta} \xi_k g_{(W_k, Y_k)}(z).$$

We will also need to consider the soft local time up to a random time $T \leq T_\Delta$:

$$(4.9) \quad G_T(z) = \sum_{k=1}^T \xi_k g_{(W_k, Y_k)}(z).$$

Analogously, we define for any deterministic time $n \geq 1$

$$(4.10) \quad G_n(z) = \sum_{k=1}^n \xi_k g_{(W_k, Y_k)}(z).$$

We denote by z_k the piece of trajectory randomly selected by the k -th soft local time, G_k .

As we have seen before, in order to simulate the random interlacements set at level u in A_1 , we actually need a

$$N_V^u \stackrel{d}{=} \text{Poisson}(u \text{cap}(V))$$

number of random walk trajectories, each started at a point in V distributed as $\bar{e}_V(\cdot)$. For $j = 1, \dots, N_V^u$ we let $((W_k^j, Y_k^j))_{k \geq 1}$ be a clothesline process started at W_1^j , so that $((W_k^j, Y_k^j))_{k \geq 1}$ is independent from $((W_k^i, Y_k^i))_{k \geq 1}$ for $i \neq j$, and so that W_1^j is distributed as $\bar{e}_V(\cdot)$. Let T_Δ^j be the killing time associated with $((W_k^j, Y_k^j))_{k \geq 1}$. We denote by

$$(4.11) \quad G^j(z) = \sum_{k=1}^{T_\Delta^j} \xi_k^j g_{(W_k^j, Y_k^j)}(z)$$

the soft local time associated with the j -th clothesline process. It should be clear from Proposition 3.2 that we can simulate all the random elements $(\sigma_{W_k^j, Y_k^j})_{j, k \geq 1}$ at the same time using only one realization of a Poisson point process in $\Sigma \times \mathbb{R}_+$. As the Corollary 3.3 shows, in order to control the values our random elements take we only need to control the function

$$(4.12) \quad G_u^\Sigma(z) = \sum_{j=1}^{N_V^u} G^j(z),$$

the soft local time associated with the whole process. With such objective in mind we for now set our goals at estimating the soft local time's moments. We first show an easier way to express the expectation of $G(z)$.

Proposition 4.1. *Using the same notation as above, we have*

$$(4.13) \quad \mathbb{E}(G(z)) = \mathbb{E}\left(\sum_{k=1}^{T_\Delta} 1_{\{\Xi(X_{D_{k-1}}, X_{R_k}) = \Xi(z)\}}\right).$$

Proof. In fact,

$$(4.14) \quad \begin{aligned} \mathbb{E}(G(z)) &= \mathbb{E}\left(\sum_{k=1}^{T_\Delta} g_{(W_k, Y_k)}(z)\right) = \mathbb{E}\left(\sum_{k=1}^{T_\Delta} \mathbb{P}_{W_k, Y_k}[\Xi(W_k, Y_k) = \Xi(z)]\right) \\ &= \mathbb{E}\left(\sum_{k=1}^{T_\Delta} 1_{\{\Xi(W_k, Y_k) = \Xi(z)\}}\right) = \mathbb{E}\left(\sum_{k=1}^{T_\Delta} 1_{\{\Xi(X_{D_{k-1}}, X_{R_k}) = \Xi(z)\}}\right). \end{aligned}$$

□

We have then that the expectation of $G(z)$, for $z \neq \Theta$, is the same as the expectation of how many times a random walk started at W_1 will do an excursion on A_2^C with starting and ending points given by $\Xi(z)$.

It is clear that the same computation works for any starting distribution for W_1 . Given $y \in \partial A_2$, we let $\beta_y(\cdot)$ be the hitting measure on V of a simple random walk started at y . We are then able to take $\beta_y(\cdot)$ as the starting distribution of W_1 . Let then \mathcal{P}_{β_y} be the global process's measure in which the clothesline process's starting distribution is given by $\beta_y(\cdot)$, and let \mathbb{E}_{β_y} be its associated expectation. We are then required to allow the clothesline process to start at the cemetery state Δ , denoting the failure of the random walk trajectory started at y to reach V . In an analogous definition, we let \mathcal{P}_{w_0} be the global process's measure with $w_0 \in V$ as the clothesline process's starting point, and let \mathbb{E}_{w_0} be its associated expectation.

The next proposition, adapted from Theorem 4.8 of [13], gives a bound on the second moment $\mathbb{E}(G(z))^2$.

Proposition 4.2. *For any $w_0 \in V$,*

$$(4.15) \quad \mathbb{E}_{w_0}(G(z))^2 \leq 2\mathbb{E}_{w_0}(G(z)) \left(\sup_{w' \in V} \mathbb{E}_{w'} G(z) + \sup_{w,y} g_{(w,y)}(z) \right).$$

Proof. Recall that the second moment of a $\text{Exp}(1)$ random variable equals 2. For $z \in \Sigma$ and $n \geq 1$, we write

$$\begin{aligned} \mathbb{E}_{w_0}(G_n(z))^2 &= \mathbb{E}_{w_0} \left(\sum_{k=1}^n \xi_k g_{(W_k, Y_k)}(z) \right)^2 \\ &= \mathbb{E}_{w_0} \left(\sum_{k=1}^n \xi_k^2 g_{(W_k, Y_k)}^2(z) \right) + \mathbb{E}_{w_0} \left(2 \sum_{k < k' \leq n} \xi_k \xi_{k'} g_{(W_k, Y_k)}(z) g_{(W_{k'}, Y_{k'})}(z) \right) \\ &\leq \sum_{k=1}^n \mathbb{E} \xi_k^2 \sup_{w,y} g_{(w,y)}(z) \mathbb{E}_{w_0} g_{(W_k, Y_k)}(z) + 2 \sum_{k=1}^{n-1} \sum_{k'=k+1}^n \mathbb{E}_{w_0} (g_{(W_k, Y_k)}(z) g_{(W_{k'}, Y_{k'})}(z)) \\ &\leq 2 \sup_{w,y} g_{(w,y)}(z) \mathbb{E}_{w_0} G_n(z) + 2 \sum_{k=1}^{n-1} \sum_{k'=k+1}^n \mathbb{E}_{w_0} (g_{(W_k, Y_k)}(z) \mathbb{E}_{w_0} (g_{(W_{k'}, Y_{k'})}(z) \mid W_k, Y_k)) \\ &\leq 2 \sup_{w,y} g_{(w,y)}(z) \mathbb{E}_{w_0} G_n(z) + 2 \sum_{k=1}^{n-1} \mathbb{E}_{w_0} \left(g_{(W_k, Y_k)}(z) \mathbb{E}_{\beta_{Y_k}} \left(\sum_{m=1}^{n-k} g_{(W_m, Y_m)}(z) \right) \right) \\ &\leq 2 \sup_{w,y} g_{(w,y)}(z) \mathbb{E}_{w_0} G_n(z) + 2 \sup_{w'} \mathbb{E}_{w'} \left(\sum_{m=1}^{n-k} g_{(W_m, Y_m)}(z) \right) \mathbb{E}_{w_0} \left(\sum_{k=1}^{n-1} g_{(W_k, Y_k)}(z) \right) \\ &\leq 2 \mathbb{E}_{w_0}(G_n(z)) \left(\sup_{w'} \mathbb{E}_{w'} G_n(z) + \sup_{w,y} g_{(w,y)}(z) \right), \end{aligned}$$

so that the result is proved for time n . Letting n go to infinity, by the monotone convergence theorem we can prove the result for the stopping time T_Δ . \square

For this paper's results, an estimate on the exponential moments of G will be essential. The next proposition, again adapted from [13] (propositions 4.3 and 4.2 are proved in the context of Markov chains in the original paper), gives us such an estimate.

Proposition 4.3. *Given $\hat{z} \in \Sigma$ and measurable $\Gamma \subset \Sigma$, let*

$$(4.16) \quad \begin{aligned} \alpha &= \inf \left\{ \frac{g_{(w,y)}(z')}{g_{(w,y)}(\hat{z})}; (w,y) \in V \times \partial A_2, z' \in \Gamma, \hat{z} \in \mathcal{K} \right\}, \\ N(\Gamma) &= \#\{k \leq T_\Delta; z_k \in \Gamma\}, \text{ and} \\ \ell &\geq \sup_{(w,y) \in V \times \partial A_2} g_{(w,y)}(\hat{z}). \end{aligned}$$

Then, for any $v \geq 2$,

$$(4.17) \quad \mathcal{P}[G(\hat{z}) \geq v\ell] \leq \mathcal{P}[G(\hat{z}) \geq \ell] \left(\exp \left\{ - \left(\frac{v}{2} - 1 \right) \right\} + \sup_{w'} \mathcal{P}_{w'} [\eta(\Gamma \times [0, \frac{1}{2}v\ell\alpha]) \leq N(\Gamma)] \right)$$

(note that $\eta(\Gamma \times [0, \frac{1}{2}v\ell\alpha])$ is a random variable with distribution $\text{Poisson}(\frac{1}{2}v\ell\alpha\mu(\Gamma))$).

The number $\alpha = \alpha(\Gamma)$ above gives us a regularity condition: whenever α is uniformly larger than some constant $c > 0$, we have that the density function $g_{(w,y)}(\cdot)$ when restricted to the subset Γ cannot vary too much.

We first explain the intuition behind the terms in the right-hand side of (4.17). The first term in the product is explained by the fact that in order for $G(\hat{z})$ to get past $v\ell$, it must first overcome ℓ . The first summand inside the parenthesis corresponds to the probability that the sum $G(\hat{z})$ overcomes ℓ at the same “time” it overcomes $v\ell 2^{-1}$, that is, a overshooting probability. The second summand corresponds to a large deviation estimate, and generally, as v grows, $N(\Gamma)$ becomes much smaller than the expected value of $\eta(\Gamma \times [0, \frac{1}{2}v\ell\alpha])$.

Proof. We define the stopping time (with respect to the filtration $\mathcal{F}_n = \sigma((W_k, Y_k), \xi_k, k \leq n)$)

$$(4.18) \quad T_\ell = \inf\{k \geq 1; G_k(\hat{z}) \geq \ell\}.$$

For $v \geq 2$, we have

$$(4.19) \quad \begin{aligned} & \mathcal{P}[G(\hat{z}) \geq v\ell] \\ & \leq \mathcal{P}[T_\ell < \infty, G_{T_\ell}(\hat{z}) \geq \frac{v}{2}\ell] + \mathcal{P}[T_\ell < \infty, G_{T_\ell}(\hat{z}) < \frac{v}{2}\ell, G(\hat{z}) - G_{T_\ell}(\hat{z}) > \frac{v}{2}\ell] \end{aligned}$$

(note that $\mathcal{P}[G(\hat{z}) \geq \ell] = \mathcal{P}[T_\ell < \infty]$). We first estimate the first term in the right side of the above inequality. By the memoryless property of the exponential distribution, we have

$$(4.20) \quad \begin{aligned} & \sum_{n \geq 1} \mathbb{E} \left(G_{n-1}(\hat{z}) < \ell, \mathcal{P}[\xi_n g_{(W_n, Y_n)}(\hat{z}) > \frac{v}{2}\ell - G_{n-1}(\hat{z}) \mid W_{n-1}, Y_{n-1}, G_{n-1}] \right) \\ & \leq \sum_{n \geq 1} \mathbb{E} \left(G_{n-1}(\hat{z}) < \ell, \mathcal{P}[\xi_1 g_{(W_n, Y_n)}(\hat{z}) > \ell - G_{n-1}] \mathcal{P}[\xi_1 g_{(W_n, Y_n)}(\hat{z}) > (\frac{v}{2} - 1)\ell] \right) \\ & \leq \mathcal{P}[T_\ell < \infty] \sup_{(w', y')} \mathcal{P}[\xi_1 g_{(w', y')}(\hat{z}) > (\frac{v}{2} - 1)\ell] \\ & \leq \mathcal{P}[T_\ell < \infty] \exp \left\{ - \left(\frac{v}{2} - 1 \right) \right\}. \end{aligned}$$

Now, to bound the second term in the right side of (4.19), we write

$$(4.21) \quad \begin{aligned} & \mathbb{E}(T_\ell < \infty, G_{T_\ell}(\hat{z}) < \frac{v}{2}\ell, \mathcal{P}[G(\hat{z}) - G_{T_\ell}(\hat{z}) > \frac{v}{2}\ell \mid G_1, \dots, G_{T_\ell}]) \\ & \leq \mathcal{P}[T_\ell < \infty] \sup_{w'} \mathcal{P}_{w'}[G(\hat{z}) > \frac{v}{2}\ell]. \end{aligned}$$

Using that for any $z' \in \Sigma$

$$(4.22) \quad G(z') = \sum_{k=1}^{T_\Delta} \xi_k g_{(W_k, Y_k)}(z') \geq \sum_{k=1}^{T_\Delta} \alpha \xi_k g_{(W_k, Y_k)}(\hat{z}) 1_\Gamma(z') = \alpha G(\hat{z}) 1_\Gamma(z').$$

we obtain, for all z' ,

$$(4.23) \quad \begin{aligned} \mathcal{P}[G(\hat{z}) \geq \frac{v}{2}\ell] & \leq \mathcal{P}\left[G(z') \geq \frac{1}{2}v\ell\alpha, \text{ for every } z' \in \Gamma\right] \\ & \leq \mathcal{P}[\eta(\Gamma \times [0, \frac{1}{2}v\ell\alpha]) \leq N(\Gamma)]. \end{aligned}$$

Collecting (4.19), (4.20), (4.21) and (4.23) we finish the proof of the result. \square

5. CONDITIONAL DECOUPLING

We begin this section gathering some facts needed for the proof of the main theorem of this paper. But first we give an overview of main argument presented in this section. We will simulate the random interlacements set intersected with A_1 in two ways. In the first way we will simulate $\mathcal{I}_{A_1}^u$ using G_u^Σ , that is, we will simulate $\mathcal{I}_{A_1}^u$ using the soft local times indexed by the clothesline processes. In the second way, we will construct a set made up from random walk trajectories in A_1 in a similar way to the construction of $\mathcal{I}_{A_1}^u$,

the only difference will be that the soft local times used in this second construction will be indexed by a given nonrandom sequence $\hat{\zeta}$ of pairs of points belonging $V \times \partial A_2$. We will denote this second random set by $\mathcal{I}_{A_1|\hat{\zeta}}^u$, and we will show using the soft local times method that $\mathcal{I}_{A_1|\hat{\zeta}}^u$ and $\mathcal{I}_{A_1}^u$ are usually very similar to each other. We then prove a similar result when the pairs of points that constitute the nonrandom sequence all belong to the boundary of a set contained in A_2 .

Throughout this section we will again only differentiate between A_1° and A_1^\square when the need arises. We start by stating the following bound

$$(5.1) \quad \sup_{\substack{w' \in V \\ y' \in \partial A_2}} \mathbb{P}_{w', y'} [\Xi(w', y') = (w_0, y_0)] \leq cs^{-2(d-1)},$$

for which the proof is technical and we thus postpone it to subsection A.1 of the appendix.

Let $z \in \Sigma$ be such that $\Xi(z) = (w_0, y_0)$, and let $h := \text{dist}(w_0, y_0)$. We let $F(w_0, y_0)$ stand for $G(z)$, making explicit the dependence of the soft local time on the endvertices $\Xi(z)$. We define

$$(5.2) \quad \pi(w_0, y_0) := \mathbb{E}(F(w_0, y_0)).$$

We define $f_{A_1}(w_0, y_0)$ to be the probability that the simple random walk started at w_0 visits y_0 before hitting A_2 . We will prove in the appendix (see Section A.1, propositions A.2 and A.3) the following bounds for these probabilities:

(i) Given $(w_0, y_0) \in A_1^\circ \times V^\circ$, there are constants $c_1, c_2 > 0$ such that

$$(5.3) \quad c_1 \frac{s^2}{h^d} \leq f_{A_1^\circ}(w_0, y_0) \leq c_2 \frac{s^2}{h^d}.$$

(ii) Let $(w_0, y_0) \in A_1^\square \times V^\square$, and recall the definition of \mathfrak{H}_{r+2s} , the unsmoothed version of $A_2^{\square C}$. Let \mathfrak{H}_i^{d-1} ; $i = 1, \dots, 2d$; denote the $(d-1)$ -dimensional hyperfaces of \mathfrak{H}_{r+2s} , and let $l_i^{w_0} := \min\{\text{dist}(w_0, \mathfrak{H}_i^{d-1}), h\}$, and $l_i^{y_0} := \min\{\text{dist}(y_0, \mathfrak{H}_i^{d-1}), h\}$. Then there are constants $c_1, c_2 > 0$ such that

$$(5.4) \quad c_1 \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h^{2d}} \cdot \frac{1}{h^{d-2}} \cdot \frac{l_1^{y_0} \dots l_{2d}^{y_0}}{h^{2d}} \leq f_{A_1^\square}(w_0, y_0) \leq c_2 \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h^{2d}} \cdot \frac{1}{h^{d-2}} \cdot \frac{l_1^{y_0} \dots l_{2d}^{y_0}}{h^{2d}}.$$

The following lemma, whose proof we also postpone to the appendix (Section A.2), gives us an estimate on $\pi(w_0, y_0)$.

Lemma 5.1. *Using the notation defined above we have, for constants $c_1, c_2, c_3, c_4 > 0$:*

$$(i) \quad c_1 \text{cap}(V)^{-1} s^{-1} f_{A_1}(w_0, y_0) \leq \pi_{A_1}(w_0, y_0) \leq c_2 \text{cap}(V)^{-1} s^{-1} f_{A_1}(w_0, y_0),$$

$$(ii) \quad \mathbb{E}(F(w_0, y_0)^2) \leq c_3 \text{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0).$$

Moreover, since $\text{dist}(w_0, y_0) \geq s$, we have

$$(iii) \quad \sup_{w_0, y_0} \pi(w_0, y_0) \leq c_4 \text{cap}(V)^{-1} s^{-(d-1)}.$$

We now provide a large deviation bound for $F(w_0, y_0)$.

Lemma 5.2. *There are constants $c, c_1, c_2 > 0$ such that for every $(w_0, y_0) \in V \times \partial A_2$, we have*

$$(5.5) \quad \mathcal{P}[F(w_0, y_0) > vcs^{-2(d-1)}] \leq c_1 s^{2d-3} f_{A_1}(w_0, y_0) \text{cap}(V)^{-1} e^{-c_2 v}$$

for any $v \geq 2$ (we can also assume $c_2 \leq 1$ without loss of generality).

Proof. In the proof of this particular result it will be important for us to distinguish between the constants. We will use Proposition 4.3 for $F(w_0, y_0)$, with

$$\Gamma_{w_0, y_0} := \{(w'_0, y'_0) \in \partial A_1 \times V; \max\{\|w'_0 - w_0\|, \|y'_0 - y_0\|\} \leq c_4 s\},$$

with $0 < c_4 < 1$ defined in Section A.3 of the appendix.

Using the same notation as in Proposition 4.3, we note that (5.1) implies

$$l \leq cs^{-2(d-1)}$$

and observe that $\mu(\Gamma_{w_0, y_0}) \geq c_5 s^{2(d-1)}$ for some constant $c_5 > 0$. Also, as can be seen in Section A.3 of the appendix, we have

$$\alpha \geq c_3 > 0.$$

Chebyshev's inequality and Lemma 5.1 then imply

(5.6)

$$\mathcal{P}[T_l < \infty] \leq \mathcal{P}[F(w_0, y_0) > cs^{-2(d-1)}] \leq \frac{\pi(w_0, y_0)}{cs^{-2(d-1)}} \leq c_1 s^{2d-3} f_{A_1}(w_0, y_0) \text{cap}(V)^{-1}.$$

We denote by $N(\Gamma_{w_0, y_0})$ the number of times the simple random walk trajectory associated with $F(w_0, y_0)$ makes an excursion of the form $z' \in \Sigma$ on A_2^C such that $\Xi(z') = (w', y') \in \Gamma_{w_0, y_0}$. We also let η_{w_0, y_0} stand for the number of points of the Poisson process associated with our soft local times that belong to $\Gamma_{w_0, y_0} \times [0, \frac{1}{2}vcc_3s^{-2(d-1)}]$. We note that both definitions are consistent with Proposition 4.3 and write

$$\mathcal{P}\left[\eta_{w_0, y_0} \leq N(\Gamma_{w_0, y_0})\right] \leq \mathcal{P}\left[\eta_{w_0, y_0} \leq \frac{cc_3c_5v}{4}\right] + \mathcal{P}\left[N(\Gamma_{w_0, y_0}) \geq \frac{cc_3c_5v}{4}\right].$$

We claim that both terms in the right side of the above inequality are exponentially small in v . To see why this is true, observe that:

- η_{w_0, y_0} has Poisson distribution with parameter at least $\frac{cc_3c_5v}{2}$, and
- every time the simple random walk associated with $F(w_0, y_0)$ hits ∂A_2 , with uniform positive probability the walk never reaches Γ_{w_0, y_0} again. This way $N(\Gamma_{w_0, y_0})$ is dominated by a $\text{Geometric}(c_6)$ random variable, for some constant $c_6 < 1$.

Together with (5.6) and Proposition 4.3, this finishes the proof of the lemma. \square

Let $\Psi_{w_0, y_0}(\lambda) = \mathbb{E}(e^{\lambda F(w_0, y_0)})$ be the moment generating function of $F(w_0, y_0)$. We are going to use the bounds above to estimate Ψ_{w_0, y_0} . It is elementary to obtain that $e^t - 1 \leq t + t^2$ for $t \in [0, 1]$. With this observation in mind, we write for $0 \leq \lambda \leq \frac{c_2 s^{2(d-1)}}{2c}$, where c and c_2 are the same as in the theorem above:

$$\begin{aligned} \Psi_{w_0, y_0}(\lambda) - 1 &= \mathbb{E}(e^{\lambda F(w_0, y_0)} - 1)1_{\lambda F(w_0, y_0) \leq 1} + \mathbb{E}(e^{\lambda F(w_0, y_0)} - 1)1_{\lambda F(w_0, y_0) > 1} \\ &\leq \mathbb{E}(\lambda F(w_0, y_0) + \lambda^2 F(w_0, y_0)^2) + \mathbb{E}(e^{\lambda F(w_0, y_0)} - 1)1_{\lambda F(w_0, y_0) > 1} \\ &\leq \lambda \pi(w_0, y_0) + c_1 \lambda^2 \text{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0) + \mathbb{E}(e^{\lambda F(w_0, y_0)} - 1)1_{\lambda F(w_0, y_0) > 1} \\ &\leq \lambda \pi(w_0, y_0) + c' \lambda^2 \text{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0) + \lambda \int_{\lambda^{-1}}^{\infty} e^{\lambda y} \mathcal{P}[F(w_0, y_0) > y] dy \\ &\leq \lambda \pi(w_0, y_0) + f_{A_1}(w_0, y_0) \text{cap}(V)^{-1} \left(c' \lambda^2 s^{-2d+2} + \lambda c' s^{2d-3} \int_{\lambda^{-1}}^{\infty} \exp\left(\frac{-c_2 s^{2(d-1)} y}{2c}\right) dy \right) \\ &\leq \lambda \pi(w_0, y_0) + f_{A_1}(w_0, y_0) \text{cap}(V)^{-1} \left(c' \lambda^2 s^{-2d+2} + c' \lambda s^{-1} \exp\left(\frac{-c_2 s^{2(d-1)} \lambda^{-1}}{2c}\right) \right) \end{aligned}$$

$$(5.7) \quad \leq \lambda \pi(w_0, y_0) + c' \lambda^2 \text{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0),$$

where we used Lemma 5.1 and Lemma 5.2. Now since $e^{-t} - 1 \leq -t + t^2$ for all $t \geq 0$, we obtain for $\lambda \geq 0$

$$(5.8) \quad \Psi_{w_0, y_0}(-\lambda) - 1 \leq -\lambda \pi(w_0, y_0) + c \lambda^2 \text{cap}(V)^{-1} s^{-2d+2} f_{A_1}(w_0, y_0),$$

(the large deviation bound of Lemma 5.2 is not necessary in this case).

Observe that if $(\chi_k, k \geq 1)$ are i.i.d. random variables with common moment generating function Ψ and N is an independent Poisson random variable with parameter θ , then

$$\mathbb{E} \exp \left(\lambda \sum_{k=1}^N \chi_k \right) = e^{(\theta(\Psi(\lambda)-1))}.$$

We let $F_k(w_0, y_0)$ denote the expectation $\mathbb{E}(G^k(z))$ defined in (4.11), when $z \in \Sigma$ is such that $\Xi(z) = (w_0, y_0)$. Using Lemma 5.1 and (5.7), we have, for $N_u^V \stackrel{d}{=} \text{Poisson}(\hat{u} \text{cap}(V))$ and any $\delta > 0$

$$(5.9) \quad \begin{aligned} \mathcal{P}[G_u^\Sigma(z) \geq (1 + \delta) \hat{u} \text{cap}(V) \pi(w_0, y_0)] &= \\ &= \mathcal{P} \left[\sum_{k=1}^{N_u^V} F_k(w_0, y_0) \geq (1 + \delta) \hat{u} \text{cap}(V) \pi(w_0, y_0) \right] \\ &\leq \frac{\mathbb{E}(\exp(\lambda \sum_{k=1}^{N_u^V} F_k(w_0, y_0)))}{\exp(\lambda(1 + \delta) \hat{u} \text{cap}(V) \pi(w_0, y_0))} \\ &\leq \exp(-\lambda(1 + \delta) \hat{u} \text{cap}(V) \pi(w_0, y_0) + \hat{u} \text{cap}(V) (\Psi_{w_0, y_0}(\lambda) - 1)) \\ &\leq \exp(-(\lambda \delta \hat{u} \text{cap}(V) \pi(w_0, y_0) - c' \lambda^2 \hat{u} s^{-2d+2} f_{A_1}(w_0, y_0))) \\ &\leq \exp(-(\lambda \delta \hat{u} c s^{-1} f_{A_1}(w_0, y_0) - c' \lambda^2 \hat{u} s^{-2d+2} f_{A_1}(w_0, y_0))). \end{aligned}$$

Analogously, with (5.8) instead of (5.7), we obtain

$$(5.10) \quad \mathcal{P}[G_u^\Sigma(z) \leq (1 - \delta) \hat{u} \text{cap}(V) \pi(w_0, y_0)] \leq \exp(-(\lambda \delta \hat{u} c s^{-1} - c' \lambda^2 \hat{u} s^{-2d+2}) f_{A_1}(w_0, y_0)).$$

We choose $\lambda = c_7 \delta s^{2d-3}$ with c_7 small enough so that $\lambda \leq \frac{c_2 s^{2(d-1)}}{2c}$, and observe that the bounds for $f_{A_1}(w_0, y_0)$ given in (5.3) and (5.4) imply

$$\begin{aligned} \inf_{w_0, y_0} f_{A_1^\square}(w_0, y_0) &\geq c s^{2d} r^{-3d+2}, \\ \inf_{w_0, y_0} f_{A_1^\circ}(w_0, y_0) &\geq c s^2 r^{-d}. \end{aligned}$$

Recall the definition of $b_{A_1^\circ}$, a number such that

$$1 \leq b_{A_1^\circ} < \frac{2d-2}{d},$$

and the definition of $b_{A_1^\square}$, a number such that

$$1 \leq b_{A_1^\square} < \frac{4d-4}{3d-2}.$$

Recall that $r \asymp s^{b_{A_1}}$. Then there exist constants $a_{A_1^\circ} = 2d - 2 - db_{A_1^\circ} > 0$ and $a_{A_1^\square} = 4d - 4 - 3db_{A_1^\square} + 2b_{A_1^\square} > 0$ such that

$$\mathcal{P}[G_u^\Sigma(z) \geq (1 + \delta)\hat{u} \text{cap}(V)\pi(w_0, y_0)] \leq \exp(-c\delta^2 \hat{u} s^{a_{A_1}}).$$

Using the union bound (note that $\partial A_1 \times V$ has $O(r^{2(d-1)})$ elements),

$$\begin{aligned} \mathcal{P}[(1 - \delta)\hat{u} \text{cap}(V)\pi(\Xi(z)) \leq G_u^\Sigma(z) \leq (1 + \delta)\hat{u} \text{cap}(V)\pi(\Xi(z)), \text{ for all } z \in \mathcal{K}] &\geq \\ (5.11) \qquad \qquad \qquad &\geq 1 - cr^{2(d-1)} \exp(-c'\delta^2 \hat{u} s^{a_{A_1}}). \end{aligned}$$

Observe that we can suppose $c' \leq 1$ without loss of generality. We define the interval

$$I_{\hat{u},z}^\delta := [(1 - \delta)\hat{u} \text{cap}(V)\pi(\Xi(z)), (1 + \delta)\hat{u} \text{cap}(V)\pi(\Xi(z))]$$

and the event

$$D_u^\delta := \{G_u^\Sigma \in I_{\hat{u},z}^\delta \text{ for all } z \in \mathcal{K}\}.$$

Using (5.11) and the union bound we obtain, for $\varepsilon > 0$ sufficiently small,

$$\mathcal{P}[D_u^{\varepsilon/4}, D_{u(1-\varepsilon)}^{\varepsilon/4}, D_{u(1+\varepsilon)}^{\varepsilon/4}] \geq 1 - cr^{2(d-1)} \exp(-c'\varepsilon^2 u s^a).$$

Since $r \asymp s^{b_{A_1}}$, by replacing the constants c and c' in the above equation we obtain

$$(5.12) \qquad \mathcal{P}[D_u^{\varepsilon/4}, D_{u(1-\varepsilon)}^{\varepsilon/4}, D_{u(1+\varepsilon)}^{\varepsilon/4}] \geq 1 - c \exp(-c'\varepsilon^2 u s^a).$$

We have just proved that with high probability, the soft local time associated to each of the processes $\mathcal{I}_{A_1}^u, \mathcal{I}_{A_1}^{u(1-\varepsilon)}$ and $\mathcal{I}_{A_1}^{u(1+\varepsilon)}$ stays confined between the graphs of two explicit deterministic functions. This happened when we let the “information” given by $\mathcal{I}_{A_2}^u$; namely the points of entrance at V and exit at ∂A_2 of the excursions on A_1 of the simple random walk trajectories of the interacements process at level u ; to be distributed according to the right law, that is, the law of the clothesline processes. When we “average” those points according to these laws we obtain a good concentration for the whole function G_u^Σ , but our goal is to obtain a similar concentration when these points are deterministic. The heuristic argument is that when something happens with high probability in the annealed law, then most of the times it will also happen with high probability in the quenched law. We will introduce some new notation to make this argument rigorous and prove our main theorem.

Given any two finite sets $K_1, K_2 \subset \mathbb{Z}^d$, not necessarily disjoint, we want to describe a collection of generalized clothesline processes between K_1 and K_2 associated with the interacements process at level u . We construct an infinite family $(X_k^{(j)}, k \geq 0)_{0 < j < \infty}$ of independent simple random walks with starting point distributed according to the normalized harmonic measure on K_1 , as we did in definition (2.2). We let $\tau_0^j \equiv 0$ and define inductively

$$\begin{aligned} \tau_{k+1}^j &:= 1\{X_{\tau_k^j}^{(j)} \in K_1\} \inf\{t > \tau_{k+1}^j; X_t^{(j)} \in K_2\} \\ &\quad + 1\{X_{\tau_k^j}^{(j)} \in K_2\} \inf\{t > \tau_{k+1}^j; X_t^{(j)} \in K_1\}, \end{aligned}$$

where $1\{\cdot\}$ denotes the indicator function of an event. We also define the random time

$$T_j := \inf_{k \geq 0} \{\tau_{k+1}^j = \infty\}.$$

We let yet again $N_u^{K_1} \stackrel{d}{=} \text{Poisson}(u \text{cap}(K_1))$ be a random variable independent from

$$(X_k^{(j)}, k \geq 0)_{0 < j < \infty}.$$

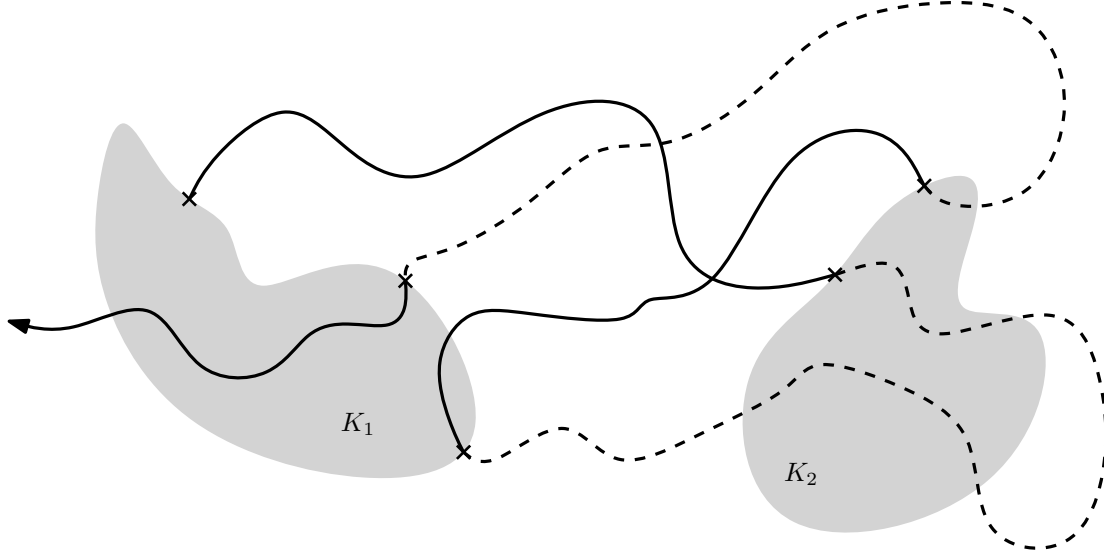


FIGURE 9. The generalized clothesline process between K_1 and K_2 , here represented by the X marks.

We then define the interlacements' clothesline processes between K_1 and K_2 at level u by

$$\text{Cloth}_u(K_1, K_2) := \left\{ \left(X_{\tau_k^j}^{(j)} \right)_{k=0}^{T_j} \right\}_{j=1}^{N_u^{K_1}}.$$

When $K_1 = V$ and $K_2 = \partial A_2$, we have

$$\text{Cloth}_u(V, \partial A_2) \stackrel{d}{=} \left\{ \left(W_k^j, Y_k^j \right)_{k=1}^{T_\Delta^j} \right\}_{j=1}^{N_u^V}.$$

We define

$$\left(\mathcal{S}_u(K_1, K_2), \sigma_u(K_1, K_2), \mathbb{P}_{K_1, K_2}^u \right)$$

to be the probability space in which $\text{Cloth}_u(K_1, K_2)$ is defined, and in which $\sigma_u(K_1, K_2)$ is the smallest σ -field in which $\text{Cloth}_u(K_1, K_2)$ is measurable. If $\hat{\zeta} \in \mathcal{S}_u(V, \partial A_2)$ and $\mathbb{P}_{V, \partial A_2}^u(\hat{\zeta}) > 0$, then we can write $\hat{\zeta}$ as a finite collection of finite sequences of points belonging to V and ∂A_2

$$\hat{\zeta} := \{\hat{\zeta}_1, \dots, \hat{\zeta}_K\},$$

where for each $j = 1, \dots, K$; $\hat{\zeta}_j$ is a finite sequence alternating between points of V and ∂A_2 . In other words, $\hat{\zeta}_j$ is a possible realization of a clothesline process. We write

$$\hat{\zeta}_j := (\zeta_0^j, \dots, \zeta_{n(j)}^j),$$

where $n(j)$ is odd, every even entry belongs to V and every odd entry belongs to ∂A_2 . We then define the soft local time associated with $\hat{\zeta}$. Using the same realization of the Poisson point process on $\Sigma \times \mathbb{R}_+$ defined on Section 4, we construct the soft local times

$$G^{\hat{\zeta}_j}(z) := \sum_{k=0}^{\frac{n(j)+1}{2}} \tilde{\zeta}_k^j g_{(\zeta_{2k}^j, \zeta_{2k+1}^j)}(z),$$

where $\tilde{\xi}_k^j$ is an exponential random variable defined in the manner of (4.8). We then define

$$G^{\hat{\zeta}}(z) := \sum_{j=1}^K G^{\hat{\zeta}^j}(z).$$

This function should be viewed as a quenched version of the soft local times G_u^Σ , when the collection of clothesline processes $\{(W_k^j, Y_k^j)_{k=1}^{T_u^j}\}_{j=1}^{N_u^V}$ is given by the deterministic element $\hat{\zeta}$. We denote by $\mathcal{I}_{A_1|\hat{\zeta}}^u$ the interlacements process inside A_1 determined by the ranges of the excursions of Σ below $G^{\hat{\zeta}}$. $\mathcal{I}_{A_1|\hat{\zeta}}^u$ is distributed as the random interlacements process inside A_1 when its associated random walks excursions have entrance points at V and exit points at ∂A_2 given by $\hat{\zeta}$. The next proposition implies that $G^{\hat{\zeta}}$ is usually between $G_{u(1-\varepsilon)}^\Sigma$ and $G_{u(1+\varepsilon)}^\Sigma$ with high probability.

Proposition 5.3. *There exists a set $\mathcal{A} \in \sigma_u(V, \partial A_2)$ such that*

$$\mathbb{P}_{V, \partial A_2}^u[\mathcal{A}] \geq 1 - \exp\left(-\frac{c'}{2}\varepsilon^2 u s^{a_{A_1}}\right),$$

and for all fixed $\hat{\zeta} \in \mathcal{A}$,

$$\begin{aligned} \mathcal{P}[G_{u(1-\varepsilon)}^\Sigma(z) \leq G^{\hat{\zeta}}(z) \leq G_{u(1+\varepsilon)}^\Sigma(z) \text{ for all } z \in \mathcal{K}] \\ \geq 1 - c \exp\left(-\frac{c'}{2}\varepsilon^2 u s^{a_{A_1}}\right). \end{aligned}$$

Proof. Observe that (5.12) implies

$$\begin{aligned} \int \mathcal{P}[G_{u(1-\varepsilon)}^\Sigma(z) \leq G^{\hat{\zeta}}(z) \leq G_{u(1+\varepsilon)}^\Sigma(z) \text{ for all } z \in \mathcal{K}] \mathbb{P}_{V, \partial A_2}^u[d\hat{\zeta}] \\ (5.13) \qquad \qquad \qquad \geq 1 - c \exp\left(-c'\varepsilon^2 u s^{a_{A_1}}\right). \end{aligned}$$

Let

$$\begin{aligned} \mathcal{A} := \left\{ \hat{\zeta} \in \mathcal{S}_u(V, \partial A_2) \text{ such that: } \mathcal{P}[G_{u(1-\varepsilon)}^\Sigma(z) \leq G^{\hat{\zeta}}(z) \leq G_{u(1+\varepsilon)}^\Sigma(z) \text{ for all } z \in \mathcal{K}] \right. \\ \left. \geq 1 - c \exp\left(-\frac{c'}{2}\varepsilon^2 u s^{a_{A_1}}\right) \right\}. \end{aligned}$$

Then (5.13) implies

$$\begin{aligned} \mathbb{P}_{V, \partial A_2}^u[\mathcal{A}] + \left(1 - c \exp\left(-\frac{c'}{2}\varepsilon^2 u s^{a_{A_1}}\right)\right) (1 - \mathbb{P}_{V, \partial A_2}^u[\mathcal{A}]) \\ \geq 1 - c \exp\left(-c'\varepsilon^2 u s^{a_{A_1}}\right), \end{aligned}$$

so that

$$\mathbb{P}_{V, \partial A_2}^u[\mathcal{A}] \geq 1 - \exp\left(-\frac{c'}{2}\varepsilon^2 u s^{a_{A_1}}\right).$$

This finishes the proof of the proposition. \square

Proposition 5.3 implies that, for $\hat{\zeta} \in \mathcal{A}$, there exists a process $(\hat{\mathcal{I}}_{A_1}^u, u \geq 0)$ distributed as the random interlacements set intersected with A_1 , and a coupling \mathcal{P} such that, for all $\varepsilon > 0$ sufficiently small and $r > 0$ sufficiently big, we have

$$(5.14) \qquad \mathcal{P}[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\zeta}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)}] \geq 1 - c \exp\left(-\frac{c'}{2}\varepsilon^2 u s^{a_{A_1}}\right).$$

To complete the proof of our main theorem we need to show that a result similar to Proposition 5.3 remains valid under a different conditioning.

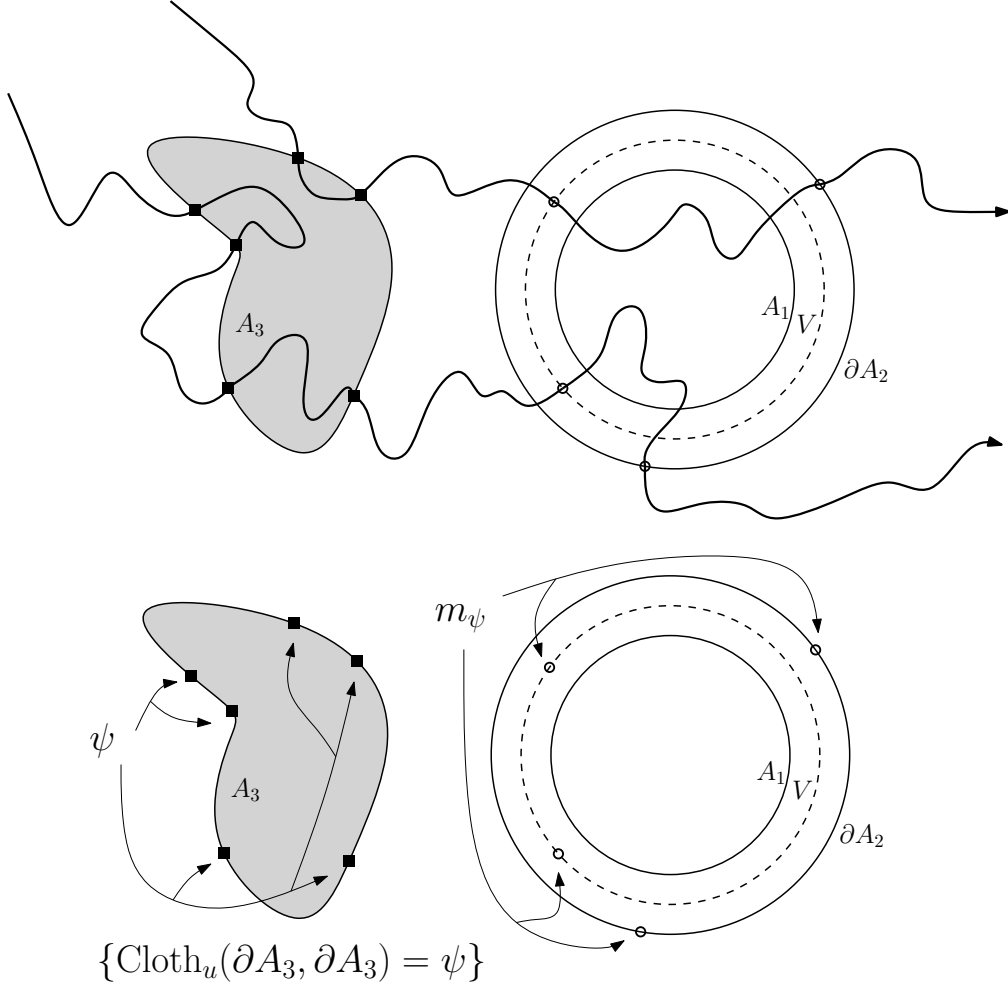


FIGURE 10. A visual representation of the random element $m_{\hat{\psi}}$.

Let $A_3 \subset A_2$ be such that $|\partial A_3| < \infty$, and write $\mathcal{I}_{A_3}^u := \mathcal{I}^u \cap A_3$. Then $\text{Cloth}_u(\partial A_3, \partial A_3)$ is well defined. Given $\hat{\psi} \in \mathcal{S}_u(\partial A_3, \partial A_3)$, we define $m_{\hat{\psi}} \equiv m_{\hat{\psi}}(\partial A_3)$ as a random element of $\mathcal{S}_u(V, \partial A_2)$ distributed as $\text{Cloth}_u(V, \partial A_2)$ conditioned on the event where the entrance and exit points at ∂A_3 of the simple random walk excursions of $\mathcal{I}_{A_3}^u$ are given by $\hat{\psi}$. We denote by $\mathcal{I}_{A_1|\hat{\psi}}^u$ the random interacements process on A_1 conditioned on the event where $\text{Cloth}_u(\partial A_3, \partial A_3)$ is equal to the deterministic element $\hat{\psi}$. Notice that all “information” given by $\mathcal{I}_{A_3}^u$ to $\mathcal{I}^u \cap A_3^C$ is contained in $\text{Cloth}_u(\partial A_3, \partial A_3)$, that is, conditioned on $\text{Cloth}_u(\partial A_3, \partial A_3)$, $\mathcal{I}_{A_3}^u$ and $\mathcal{I}^u \cap A_3^C$ are independent.

Inequality (5.14) then implies, for $\hat{\psi} \in \mathcal{S}_u(\partial A_3, \partial A_3)$,

$$\begin{aligned}
 \mathcal{P}[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\psi}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)}] &= \sum_{\hat{\zeta} \in \mathcal{S}_u(V, \partial A_2)} \mathcal{P}[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\psi}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)} \mid m_{\hat{\psi}} = \hat{\zeta}] \mathcal{P}[m_{\hat{\psi}} = \hat{\zeta}] \\
 &= \sum_{\hat{\zeta} \in \mathcal{S}_u(V, \partial A_2)} \mathcal{P}[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\zeta}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)}] \mathcal{P}[m_{\hat{\psi}} = \hat{\zeta}] \\
 &\geq \left(1 - c \exp\left(-\frac{c'}{2} \varepsilon^2 u s^{a_{A_1}}\right)\right) \mathcal{P}[m_{\hat{\psi}} \in \mathcal{A}].
 \end{aligned}
 \tag{5.15}$$

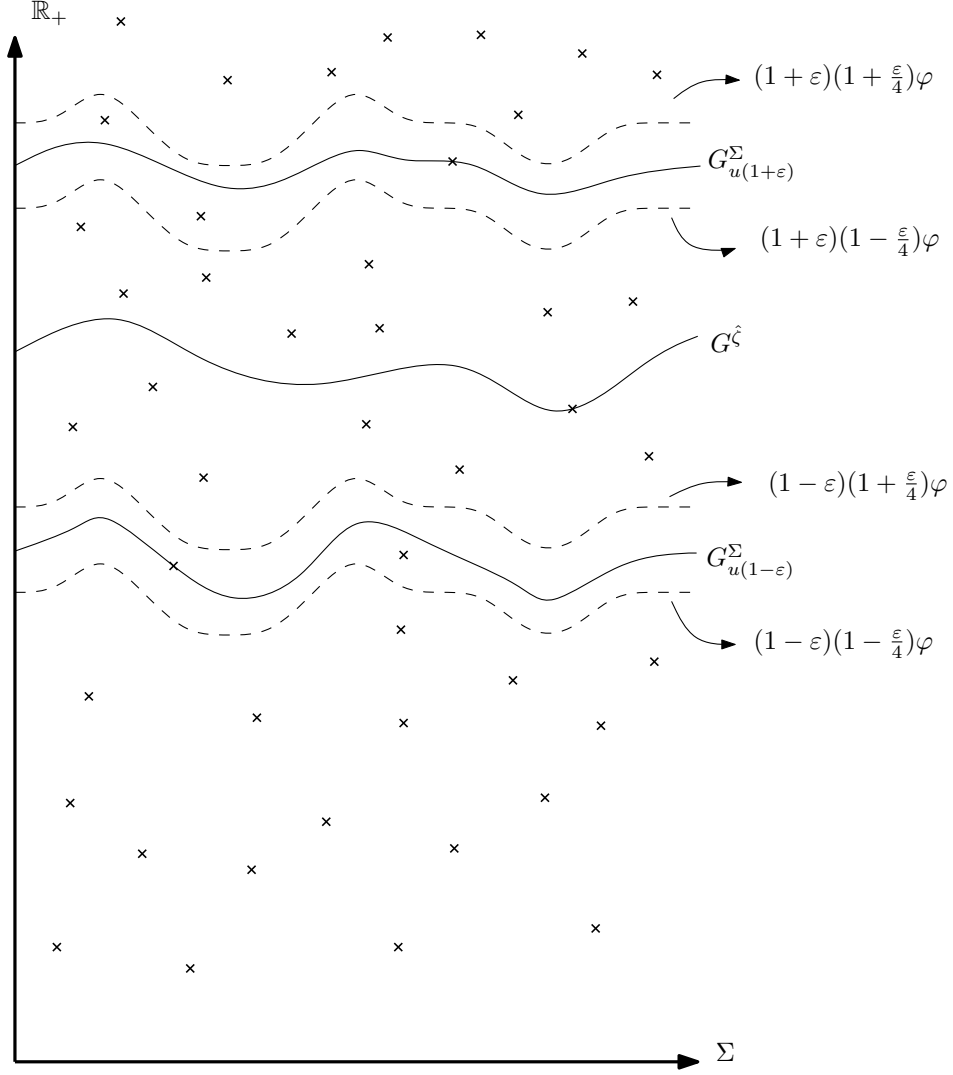


FIGURE 11. When the sequence $\hat{\zeta}$ belongs to a well behaved set \mathcal{A} , the decoupling probability is greater than $1 - c \exp\left(-\frac{c'}{2}\varepsilon^2 u s^{a_{A_1}}\right)$. The symbol φ in the figure stands for the function $u \operatorname{cap}(V) \pi(\Xi(z))$. The figure shows the decoupling event, where $G_{u(1-\varepsilon)}^{\Sigma}(z) \leq G^{\hat{\zeta}}(z) \leq G_{u(1+\varepsilon)}^{\Sigma}(z)$ for all $z \in \mathcal{K}$.

Let \mathcal{E} be the set of all $\hat{\psi} \in \mathcal{S}_u(\partial A_3, \partial A_3)$ such that

$$\mathcal{P}[m_{\hat{\psi}} \in \mathcal{A}^C] \geq \sqrt{\mathbb{P}_{V, \partial A_2}^u[\mathcal{A}^C]}.$$

Since

$$\mathbb{P}_{V, \partial A_2}^u[\mathcal{A}^C] = \int \mathcal{P}[m_{\hat{\psi}} \in \mathcal{A}^C] \mathbb{P}_{\partial A_3, \partial A_3}^u[d\hat{\psi}] \geq \mathbb{P}_{\partial A_3, \partial A_3}^u[\mathcal{E}] \sqrt{\mathbb{P}_{V, \partial A_2}^u[\mathcal{A}^C]},$$

we have

$$\mathbb{P}_{\partial A_3, \partial A_3}^u[\mathcal{E}] \leq \sqrt{\mathbb{P}_{V, \partial A_2}^u[\mathcal{A}^C]}.$$

We have proved the following theorem, which implies Theorem 2.1:

Theorem 5.4. *Using the same notation as above, we have that, for constants $c, c' > 0$, there exists a set $\mathcal{G} \in \sigma_u(\partial A_3, \partial A_3)$ such that*

$$\mathbb{P}_{\partial A_3, \partial A_3}^u[\mathcal{G}] \geq 1 - \exp\left(-\frac{c'}{4}\varepsilon^2 u s^{a_{A_1}}\right),$$

and for all $\hat{\psi} \in \mathcal{G}$,

$$(5.16) \quad \mathcal{P}[\hat{\mathcal{I}}_{A_1}^{u(1-\varepsilon)} \subseteq \mathcal{I}_{A_1|\hat{\psi}}^u \subseteq \hat{\mathcal{I}}_{A_1}^{u(1+\varepsilon)}] \geq 1 - c \exp\left(-\frac{c'}{2}\varepsilon^2 u s^{a_{A_1}}\right).$$

Moreover, for any increasing function f on the interlacements set intersected with A_1 , with $\sup |f| < M$, we have

$$(5.17) \quad \begin{aligned} (\mathbb{E}(f(\mathcal{I}_{A_1}^{u(1-\varepsilon)})) - cM \exp(-c'\varepsilon^2 u s^{a_{A_1}}))1_{\mathcal{G}} &\leq \mathbb{E}(f(\mathcal{I}_{A_1}^u) | \mathcal{I}_{A_3}^u)1_{\mathcal{G}} \\ &\leq (\mathbb{E}(f(\mathcal{I}_{A_1}^{u(1+\varepsilon)})) + cM \exp(-c'\varepsilon^2 u s^{a_{A_1}}))1_{\mathcal{G}}. \end{aligned}$$

We finish the section with a brief proof of Theorem 2.2.

Proof of Theorem 2.2. Note that, on equation (5.9), δ can be any real number greater than 0, whereas in equation (5.10), we need to have $0 < \delta < 1$. Recall that $u' > u > 0$. We have, by substituting the appropriate δ in (5.11) and ignoring the union bound term cr^{2d-2} ,

$$\begin{aligned} \mathcal{P}[G_u^\Sigma(z) < G_{u+u'}^\Sigma(z)] &\geq 1 - \mathcal{P}[G_u^\Sigma(z) > (u + u'4^{-1})\text{cap}(V)\pi(\Xi(z))] \\ &\quad - \mathcal{P}[G_{u+u'}^\Sigma(z) < 2^{-1}(u + u')\text{cap}(V)\pi(\Xi(z))] \\ &\geq 1 - \exp\left(-\frac{c}{4}(u + u')s^{a_{A_1}}\right) - \exp\left(-\frac{c}{16}\frac{(u')^2}{u^2}us^{a_{A_1}}\right) \\ &\geq 1 - \exp(-c'u's^{a_{A_1}}). \end{aligned}$$

Now, proceeding in the same manner as we did in the proof of Theorem 5.4, we are able to prove Theorem 2.2. \square

APPENDIX A. TECHNICAL ESTIMATES

A.1. Bounding the relevant probabilities. For $w_0 \in \partial A_1$ and $y_0 \in V$ we want to bound the supremum

$$(A.1) \quad \sup_{\substack{w' \in V \\ y' \in \partial A_2}} \mathbb{P}_{w', y'}[\Xi(w', y') = (w_0, y_0)]$$

from above. To do so we will bound the “hanging” probability $\mathbb{P}_{w, y}[\Xi(w, y) = (w_0, y_0)]$ for arbitrary $w \in V$ and $y \in \partial A_2$.

Given a finite nearest neighbor path γ , we denote by $|\gamma|$ its length. We will say that a path γ belongs to an event E if E occurs every time the simple random walk $(X_k, k \geq 0)$ first $|\gamma|$ steps coincide with γ . We also let $\mathbb{P}_x[\gamma]$ denote the probability that the first $|\gamma|$ steps of the simple random walk started at x coincide with γ .

In order to avoid a cumbersome notation we now introduce what, hopefully, will be a simpler way to denote our events of interest. For $w, y_0 \in V$, $w_0 \in \partial A_1$ and $y \in \partial A_2$ we define:

- $w \xrightarrow{1} w_0$: The collection of all finite nearest-neighbor trajectories starting at w that do not reach neither ∂A_1 nor ∂A_2 , except at its ending point $w_0 \in \partial A_1$. Note that this collection can be thought of as the event where the simple random walk started at w hits ∂A_1 for the first time at w_0 before reaching ∂A_2 .

- $w_0 \xrightarrow{2} y_0$: The collection of all finite nearest-neighbor trajectories starting at w_0 and ending at y_0 without reaching ∂A_2 .
- $y_0 \xrightarrow{3} y$: The collection of all finite nearest-neighbor trajectories starting at y_0 that hit ∂A_2 for the first time at y before returning to V . Note that this collection can be thought of as the event where the simple random walk started at y_0 hits ∂A_2 before returning to V and its entrance point in ∂A_2 is y .
- $w \xrightarrow{4} y$: The event where the entrance point in ∂A_2 of the simple random walk started at w is y . This event clearly can also be regarded as a collection of simple random walk trajectories starting at w and hitting ∂A_2 for the first time at y .

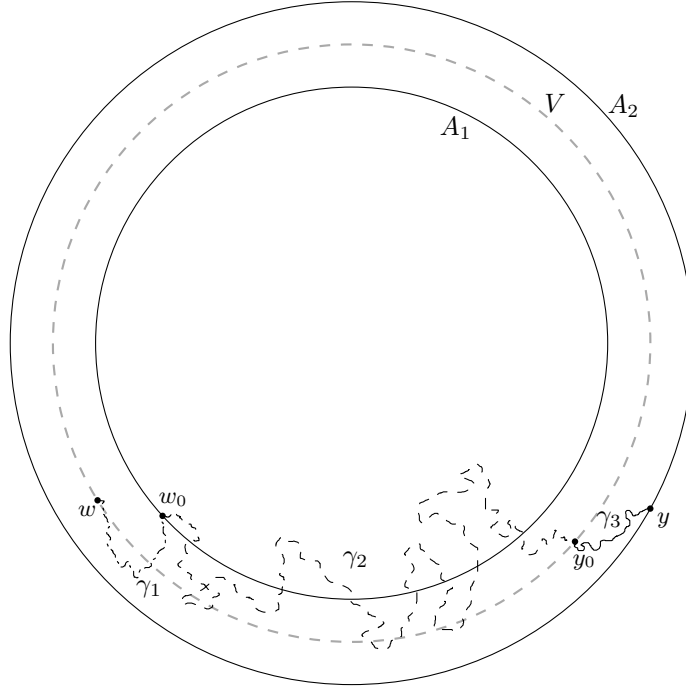


FIGURE 12. γ as the concatenation of the three paths γ_1 , γ_2 and γ_3 .

We also let $w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y$ be the “concatenation” of the first three collections, where the first trajectory’s ending point becomes the second trajectory’s starting point and so on. That is, if $\gamma \in w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y$ then γ is the concatenation of three distinct paths: $\gamma_1 \in w \xrightarrow{1} w_0$, $\gamma_2 \in w_0 \xrightarrow{2} y_0$, $\gamma_3 \in y_0 \xrightarrow{3} y$. Note that, as an event,

$$w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y = \{\Xi(w', y') = (w_0, y_0)\}.$$

With our new notation the hanging probability becomes

$$(A.2) \quad \mathbb{P}_w[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y \mid w \xrightarrow{4} y] = \frac{\mathbb{P}_w[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y]}{\mathbb{P}_w[w \xrightarrow{4} y]}.$$

We have

$$\begin{aligned}
\mathbb{P}_w[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y] &= \sum_{\gamma \in w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y} \frac{1}{2^{|\gamma|}} \\
(A.3) \qquad &= \sum_{\gamma_1 \in w \xrightarrow{1} w_0} \frac{1}{2^{|\gamma_1|}} \sum_{\gamma_2 \in w_0 \xrightarrow{2} y_0} \frac{1}{2^{|\gamma_2|}} \sum_{\gamma_3 \in y_0 \xrightarrow{3} y} \frac{1}{2^{|\gamma_3|}}.
\end{aligned}$$

Let us focus on the second sum, $\sum_{\gamma_2 \in w_0 \xrightarrow{2} y_0} \frac{1}{2^{|\gamma_2|}}$, for a moment. Each path $\gamma_2 \in w_0 \xrightarrow{2} y_0$ can be seen as the concatenation of one path γ_2^0 responsible for the walk's first visit to y_0 and a sequence of paths $\gamma_2^1, \dots, \gamma_2^k$ associated with the returns the walk makes to y_0 before hitting ∂A_2 , see Figure 13. So that

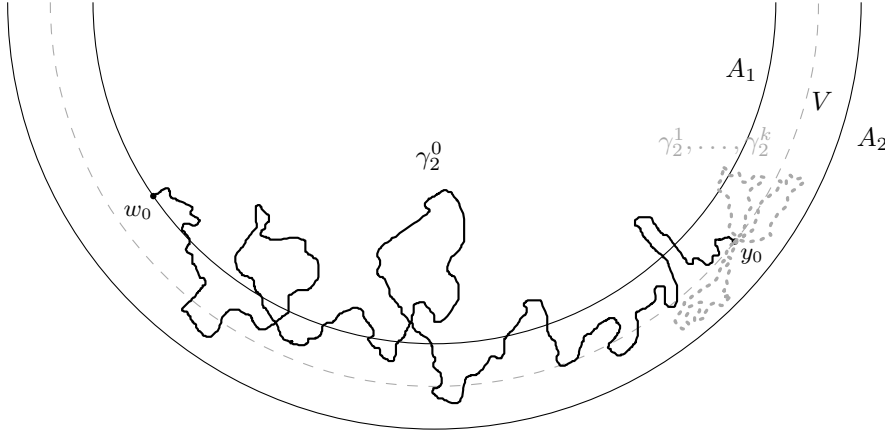


FIGURE 13. γ_2 as the concatenation of the paths $\gamma_2^0, \gamma_2^1, \dots, \gamma_2^k$.

$$(A.4) \qquad \sum_{\gamma_2} \mathbb{P}_{w_0}[\gamma_2] = \sum_{\gamma_2^0} \mathbb{P}_{w_0}[\gamma_2^0] \sum_{k \geq 1} \sum_{\gamma_2^1, \dots, \gamma_2^k} \mathbb{P}_{y_0}[\gamma_2^1] \dots \mathbb{P}_{y_0}[\gamma_2^k].$$

But for a fixed $k_0 > 0$, the last sum $\sum_{k \geq k_0} \sum_{\gamma_2^1, \dots, \gamma_2^{k_0}} \mathbb{P}_{y_0}[\gamma_2^1] \dots \mathbb{P}_{y_0}[\gamma_2^{k_0}]$ equals the probability that the simple random walk started at y_0 returns to y_0 at least k_0 times before hitting ∂A_2 . Since the walk is transient, we can use the strong Markov property to show that there exists a constant $0 < c < 1$ such that

$$(A.5) \qquad \sum_{k \geq k_0} \sum_{\gamma_2^1, \dots, \gamma_2^{k_0}} \mathbb{P}_{y_0}[\gamma_2^1] \dots \mathbb{P}_{y_0}[\gamma_2^{k_0}] < c^{k_0}.$$

We have thus shown the existence of a constant $c > 0$ such that

$$(A.6) \qquad \sum_{\gamma_2^0} \mathbb{P}_{w_0}[\gamma_2^0] \leq \sum_{\gamma_2} \mathbb{P}_{w_0}[\gamma_2] \leq c \sum_{\gamma_2^0} \mathbb{P}_{w_0}[\gamma_2^0]$$

where γ_2^0 represents any nearest neighbor path that starts at w_0 and ends at its only visit to y_0 , without ever reaching ∂A_2 . Let us update our collection's definition in view of this last computation. We denote by

- $w_0 \xrightarrow{2'} y_0$: The collection of all finite nearest-neighbor paths starting at w_0 and ending at their first visit to y_0 , without hitting ∂A_2 . This collection now can be thought of as the event where the simple random walk started at w_0 makes a visit to y_0 before hitting ∂A_2 .

Combining (A.3) with (A.6) we get

$$(A.7) \quad \mathbb{P}_w[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y] \leq c \mathbb{P}_w[w \xrightarrow{1} w_0] \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \mathbb{P}_{y_0}[y_0 \xrightarrow{3} y].$$

Our work will now reside in giving upper bounds for these three probabilities, besides giving a lower bound for $\mathbb{P}_w[w \xrightarrow{4} y]$.

There will be two results about the simple random walk we will make extensive use of. The first, which can be seen as a direct consequence of Proposition 6.5.4 of [11], essentially says that the probability that the random walk started at a distance at least h_0 from a sphere of radius h_0 enters that sphere at a specific point is of order $h_0^{-(d-1)}$, that is, the hitting measure on a sphere is comparable to the uniform distribution when the starting point of the walk is sufficiently distant. The second result is a simple application of the optional stopping theorem for submartingales and supermartingales, and can be seen in the proof of Lemma 8.5 of [13]. We state it here for the reader's convenience.

Lemma A.1. *Let $0 < \rho_1 < \rho_2$ be sufficiently large real numbers, and let $x \in B(0, \rho_2) \setminus B(0, \rho_1)$. Then*

$$(A.8) \quad \frac{|x|^{-(d-\frac{5}{2})} - (\rho_2 - 1)^{-(d-\frac{5}{2})}}{(\rho_1 + 1)^{-(d-\frac{5}{2})} - (\rho_2)^{-(d-\frac{5}{2})}} \leq \mathbb{P}_x[H_{\partial B(0, \rho_1)} < H_{\partial B(0, \rho_2)}] \leq \frac{|x|^{-(d-1)} - (\rho_2)^{-(d-1)}}{(\rho_1 - 1)^{-(d-1)} - (\rho_2)^{-(d-1)}}.$$

A.1.1. *The hanging probabilities for the ball.* In this subsection we will be concerned with the sets A_1°, V° and A_2° , and the related simple random walk probabilities.

$w \xrightarrow{1} w_0$: Let h_1 be the Euclidean distance between w and w_0 . We look to \mathbb{Z}^d as a subset of \mathbb{R}^d . Let e_1, \dots, e_d be the canonical basis of \mathbb{R}^d . Without loss of generality we assume that w and w_0 belong to the plane generated by the first vectors e_1, e_2 . If $\rho, \Phi_1, \dots, \Phi_{d-1}$ are the corresponding spherical coordinates of \mathbb{R}^d , we let, for $i_1 = 1, \dots, \lfloor \frac{2\pi r}{s} \rfloor$ and $i_k = 1, \dots, \lfloor \frac{\pi r}{s} \rfloor$, $k = 2, \dots, d-1$:

$$(A.9) \quad E_{i_1, \dots, i_{d-1}} = \left\{ (\rho, \Phi_1, \dots, \Phi_{d-1}) \in \mathbb{R}^d, r \leq \rho \leq r + 2s, \frac{(i_1-1)s}{2\pi r} \leq \Phi_1 \leq \frac{i_1 s}{2\pi r}, \right. \\ \left. \frac{(i_k-1)s}{\pi r} \leq \Phi_k \leq \frac{i_k s}{\pi r} \text{ for all } k = 2, \dots, d-1 \right\}$$

We also let C_1 be a discrete ball of radius s contained in A_1 in such a way that it intersects ∂A_1 only at w_0 . We refer any reader skeptic about the existence of such discrete ball to [13], Section 8. There is a constant $c_1 > 0$ such that the random walk started at w will have to cross at least $\lfloor \frac{c_1 h_1}{s} \rfloor$ sets of the form $E_{i_1, \dots, i_{d-1}}$ to reach w_0 . Each time the walk reaches a set $E_{i'_1, \dots, i'_{d-1}}$, the probability that it will reach another set of the form $E_{i_1, \dots, i_{d-1}}$ at distance at least s from $E_{i'_1, \dots, i'_{d-1}}$, before hitting either ∂A_1 or ∂A_2 , is bounded from above by a constant $0 < c_2 < 1$, as can be seen using Donsker's Invariance Principle (see Section 3.4 of [11]). Using the strong Markov property, we can show that the probability that the walk started at w crosses at least $\frac{c_1 h_1}{s}$ sets of the form $E_{i_1, \dots, i_{d-1}}$ before hitting $\partial A_1 \cup \partial A_2$ is smaller than $c_2^{\lfloor \frac{c_1 h_1}{s} \rfloor}$.

We note that it is harder for a walk started at some $x \in A_2^C \setminus A_1$ to hit w_0 before any other point in ∂A_1 than it is to hit w_0 before any other point in C_1 ,

$$\mathbb{P}_x[X_{H_{\partial A_1}} = w_0] \leq \mathbb{P}_x[X_{H_{C_1}} = w_0].$$

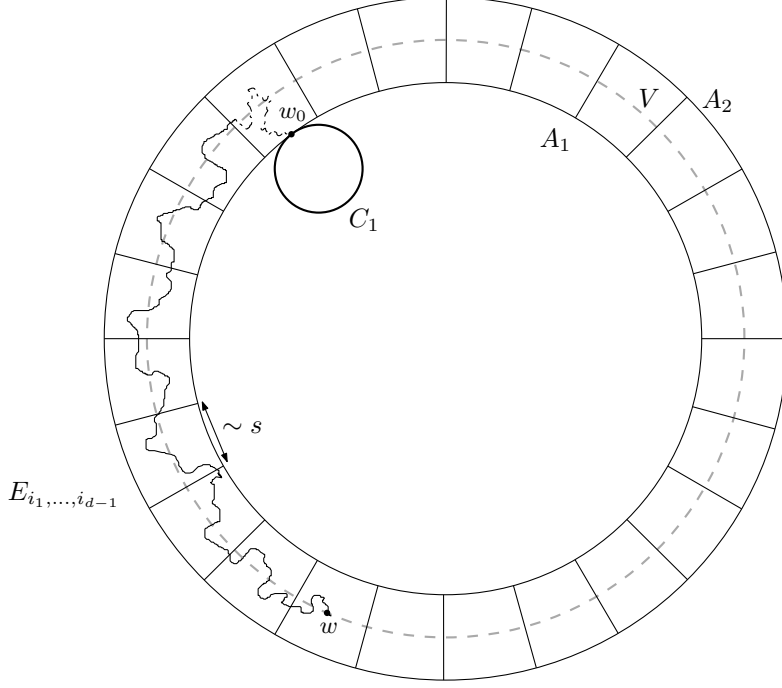


FIGURE 14. A path belonging to $w \xrightarrow{1} w_0$ has to cross $\frac{c_1 h_1}{s}$ sets of the form $E_{i_1, \dots, i_{d-1}}$ before hitting w_0 in C_1 .

We have already noted that the probability of hitting a discrete sphere of radius s at a specific point at distance of order s , is of order $s^{-(d-1)}$, as can be seen in Proposition 6.5.4 of [11]. In conjunction with last paragraph's argument, this shows the existence of constants $c_3, c_4 > 0$ such that

$$(A.10) \quad \mathbb{P}_w[w \xrightarrow{1} w_0] \leq c_4 e^{\frac{-c_3 h_1}{s}} s^{-(d-1)}.$$

$y_0 \xrightarrow{3} y$: We define $y \xrightarrow{3'} y_0$ to be the event where the walk, started at y , hits y_0 in V before reaching any other point in V or ∂A_2 . From the simple random walk's reversibility, we have

$$(A.11) \quad \mathbb{P}_y[y \xrightarrow{3'} y_0] = \mathbb{P}_{y_0}[y_0 \xrightarrow{3} y].$$

Let C_1 now be a discrete ball of radius $\frac{s}{2}$ contained in A_2 in such a way that $C_1 \cap \partial A_2 = \{y\}$, and let C_2 be a discrete ball of radius $\frac{s}{3}$ concentric with C_1 . We can use Lemma A.1 and some elementary calculus to show that if the simple random walk starts at y , the probability that it hits C_2 before hitting C_1 is of order s^{-1} . Using the strong Markov property, the argument then continues the same way as the argument for the bound for $\mathbb{P}_w[w \xrightarrow{1} w_0]$. Let h_3 be the Euclidean distance between y_0 and y . Then there are constants $c_1, c_2 > 0$ such that

$$(A.12) \quad \mathbb{P}_w[y \xrightarrow{3} y_0] \leq c_1 e^{\lfloor \frac{-c_2 h_3}{s} \rfloor} s^{-(d-1)} s^{-1}.$$

$w_0 \xrightarrow{2'} y_0$: Let h be the Euclidean distance between w_0 and y_0 . Assume $h > 20s$. Let w_1 be the point on ∂A_2 closest to w_0 . We let C_2 now be a discrete ball of radius $\frac{h}{6}$ that intersects ∂A_2 only at w_1 and lies outside of A_2^C . We let C_3 be the discrete ball of radius $\frac{h}{3}$ that is concentric with C_2 . In order for the walk started at w_0 to reach y_0

without leaving A_2^C , it first has to reach ∂C_3 before hitting C_2 . Lemma A.1 and some calculus show that the probability of such event is of order $\frac{s}{h}$.

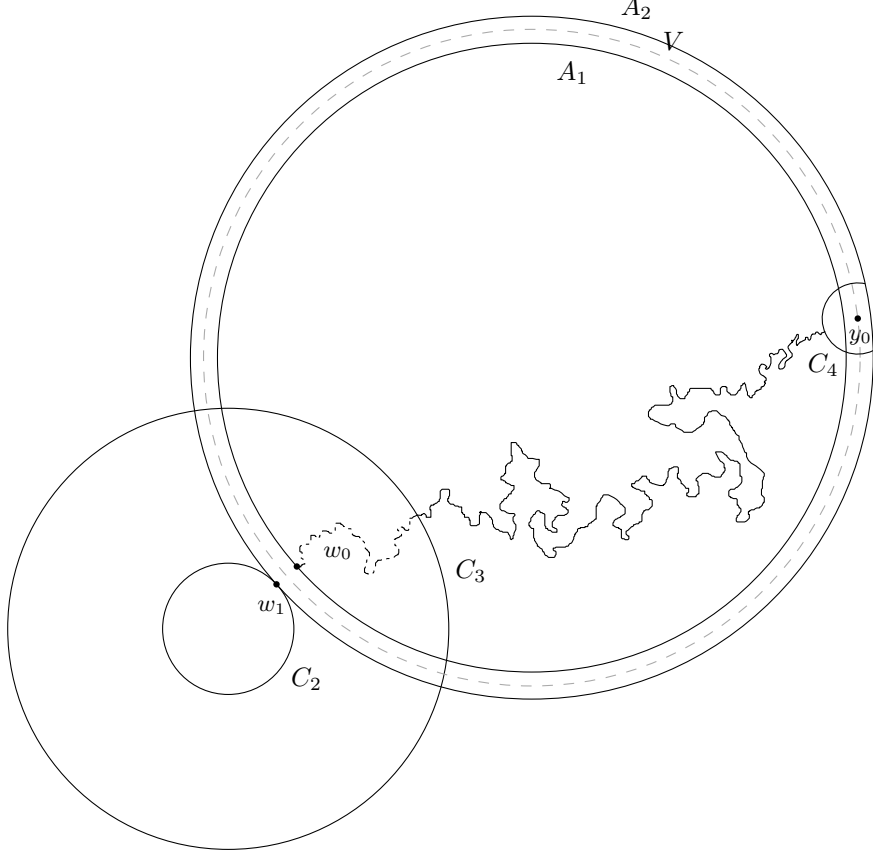


FIGURE 15. A walk started at w_0 has to reach ∂C_3 before ∂C_2 and then reach $C_4 \setminus A_2^C$ in order to reach y_0 .

In order for the walk to reach a y_0 , it has first to reach a sphere ∂C_4 of radius $3s$ centered at y_0 . Conditioned on the event where ∂C_4 is reached before the walk hits ∂A_2 , the probability that the walk reaches y_0 before reaching ∂A_2 is smaller than $cs^{-(d-2)}$, for a constant $c > 0$, as can be seen using the Green's function estimate (2.1).

Let y_1 be the point on ∂A_2 closest to y_0 . Let C_5 be a discrete ball of radius h such that the intersection $C_5 \cap \partial A_2$ has diameter $6s$ and center of mass as close as possible to y_1 . By Donsker's Invariance Principle, there is a constant $c_1 > 0$ such that a simple random walk started at any point in $\partial C_4 \cap A_2^C$ has probability at least c_1 of reaching $C_5 \cap \partial A_2$ before $\partial A_2 \setminus C_5$. Let $w_2 \in A_2^C$ be any point at distance at least $\frac{h}{2}$ from y_0 . For a simple random walk starting at w_2 we define the events:

$D_{C_5 \cap \partial A_2} := \{H_{C_5 \cap \partial A_2} \leq H_{\partial A_2 \setminus C_5}\}$; the event where the simple random walk reaches $C_5 \cap \partial A_2$ before reaching any other point in ∂A_2 .

$D_{\partial C_5 \cap A_2^C} := \{H_{\partial C_5 \cap A_2^C} \leq H_{\partial A_2 \setminus C_5}\}$; the event where the simple random walk reaches $\partial C_5 \cap A_2^C$ before reaching any other point in ∂A_2 .

$D_{C_4 \setminus A_2} := \{H_{C_4 \setminus A_2} \leq H_{\partial A_2 \setminus C_4}\}$; the event where the simple random walk reaches $C_4 \setminus A_2$ before reaching any other point in ∂A_2 .

$D_{y_0} := \{H_{y_0} \leq H_{\partial A_2}\}$; the event where the simple random walk reaches y_0 before hitting ∂A_2 .

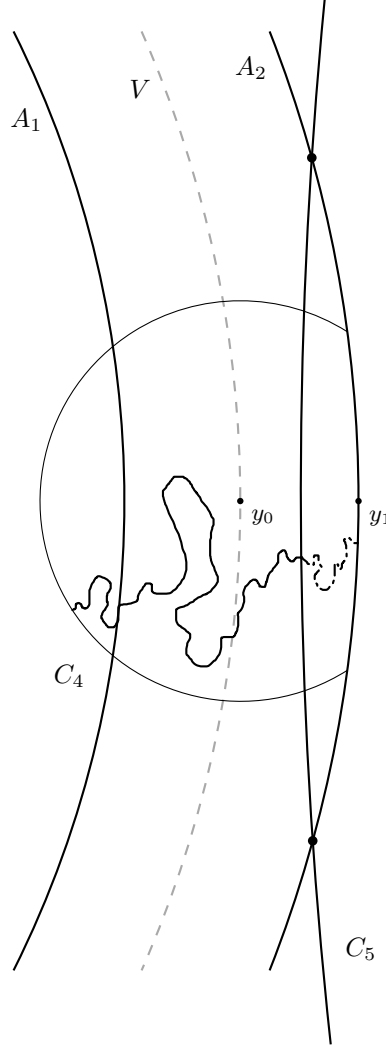


FIGURE 16. We show that, if starting at a distant point w_2 , the probability of the simple random walk hitting $C_4 \setminus A_2^C$ and the probability of hitting $C_5 \cap A_2^C$ are comparable.

From the above discussion it is clear that:

$$(A.13) \quad \mathbb{P}_{w_2}[D_{C_5 \cap \partial A_2}] \leq \mathbb{P}_{w_2}[D_{\partial C_5 \cap A_2^C}],$$

$$(A.14) \quad \mathbb{P}_{w_2}[D_{y_0}] = \mathbb{P}_{w_2}[D_{y_0} \mid D_{C_4 \setminus A_2}] \mathbb{P}_{w_2}[D_{C_4 \setminus A_2}],$$

$$(A.15) \quad \mathbb{P}_{w_2}[D_{C_4 \setminus A_2^C}] \leq \frac{1}{c_1} \mathbb{P}_{w_2}[D_{C_5 \cap \partial A_2}].$$

Using Proposition 6.5.4 of [11] we can see that there is a constant $c > 0$ such that

$$(A.16) \quad \mathbb{P}_{w_2}[D_{\partial C_5 \cap A_2^C}] \leq c \frac{s^{d-1}}{h^{d-1}}.$$

Collecting the estimates (A.13, A.14, A.15, A.16), using the strong Markov property, and bounding

$$\mathbb{P}_{w_2}[D_{y_0} \mid D_{C_4 \setminus A_2}]$$

by the Green's function estimate (2.1), we see that there is a constant $c > 0$ such that

$$(A.17) \quad \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \leq c \frac{s}{h} \cdot \frac{s^{d-1}}{h^{d-1}} s^{-(d-2)} = c \frac{s^2}{h^d}.$$

If $h < 20s$ the result follows after using Green's Function.

We also provide a lower bound for $\mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0]$, which we will need later. Suppose $h \leq \frac{r}{2}$. Let C'_3 be a discrete ball of radius $2h$ contained in A_2^C that intersects ∂A_2 only at w_1 . Let C'_2 be a discrete ball of radius $\frac{h}{2}$ concentric with C'_3 . Let us describe an event of probability greater than $c_1 \frac{s^2}{h^d}$, for some constant $c_1 > 0$, that is contained in $w_0 \xrightarrow{2'} y_0$. First the walk needs to hit $\partial C'_2$ before hitting $\partial C'_3$. The probability of such event is of order $\frac{s}{h}$, as can be seen using Lemma A.1. We will denote by w_2 the point in which the walk enters $\partial C'_2$.

We define C'_5 to be the discrete ball of radius $2h$ such that its center lies inside A_2^C and the intersection $C'_5 \cap \partial A_2$ coincides with $C_4 \cap \partial A_2$. In addition to all events defined in the proof of the upper bound for $\mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0]$, we define the event, for a simple random walk starting in the interior of C'_5 :

$D_{\partial C'_5 \setminus A_2^C} := \{H_{\partial C'_5 \setminus A_2^C} \leq H_{\partial A_2 \setminus C'_5}\}$; the event where the simple random walk started in the interior of C'_5 reaches $\partial C'_5 \setminus A_2^C$ before reaching $\partial A_2 \setminus C'_5$.

We note that w_2 is in the interior of C'_5 and that $D_{\partial C'_5 \setminus A_2^C} \subset D_{\partial C_4 \setminus A_2}$. We then have:

$$(A.18) \quad \begin{aligned} \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] &\geq \sum_{w_2 \in \partial C'_2} \mathbb{P}_{w_0}[H_{\partial C'_2} < H_{\partial C'_3}, X_{H_{\partial C'_2}} = w_2] \mathbb{P}_{w_2}[D_{y_0}] \\ &= \sum_{w_2 \in \partial C'_2} \mathbb{P}_{w_0}[H_{\partial C'_2} < H_{\partial C'_3}, X_{H_{\partial C'_2}} = w_2] \mathbb{P}_{w_2}[D_{y_0} \mid D_{C_4 \setminus A_2}] \mathbb{P}_{w_2}[D_{C_4 \setminus A_2}] \\ &\geq \sum_{w_2 \in \partial C'_2} \mathbb{P}_{w_0}[H_{\partial C'_2} < H_{\partial C'_3}, X_{H_{\partial C'_2}} = w_2] \mathbb{P}_{w_2}[D_{y_0} \mid D_{C_4 \setminus A_2}] \mathbb{P}_{w_2}[D_{\partial C'_5 \setminus A_2}]. \end{aligned}$$

Using Harnack's Principle (Theorem 6.3.9 of [11]) we are able to show the existence of a constant $c_2 > 0$ such that

$$(A.19) \quad \mathbb{P}_{w_2}[D_{\partial C'_5 \setminus A_2}] \geq c_2 \frac{s^{d-1}}{h^{d-1}}.$$

With this and (A.18) we can find a constant $c_1 > 0$ such that

$$(A.20) \quad \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \geq c_1 \frac{s^2}{h^d}.$$

If $h \geq \frac{r}{2}$ we simply replace the balls C'_3 and C'_5 by A_2^C , the ball C'_2 by a ball concentric with A_2^C but with the diameter halves, and continue the proof identically.

$w \xrightarrow{4} y$: Let w_3 be the closest point to w in ∂A_2 . Let h_4 be the Euclidean distance between w and y , and suppose $h_4 \leq \frac{r}{2}$. Let C_6 be a discrete ball of radius $2h_4$ contained in A_2^C that intersects ∂A_2 only at w_3 . Let C_7 be a discrete ball of radius $\frac{h_4}{2}$ concentric with C_6 . Then again Lemma A.1 and some calculus show that the probability that a simple random walk started at w will reach ∂C_7 before reaching ∂C_6 is less than the probability that the same walk will reach ∂C_7 before hitting ∂A_2 and bigger than $c_1 \frac{s}{h_4}$, for some constant $c_1 > 0$.

Let C_8 be a discrete ball of radius $2h_4$ contained in A_2^C that intersects ∂A_2 only at y . Let y_3 be a fixed point in ∂C_7 . Then the probability that a simple random walk started at y_3 hits y before hitting any other point in ∂C_8 is smaller than the probability that the same walk reaches y before any other point in ∂A_2 and bigger than $\frac{c_2}{h_4^{d-1}}$, for some constant $c_2 > 0$, by the Harnack's Principle (Theorem 6.3.9 of [11]) and Lemma 6.3.7 of [11]. Figure 17 illustrates the argument. Using the strong Markov property, we then have

$$(A.21) \quad \mathbb{P}_{w_0}[w \xrightarrow{4} y] \geq c \frac{s}{h_4} h_4^{-(d-1)}.$$

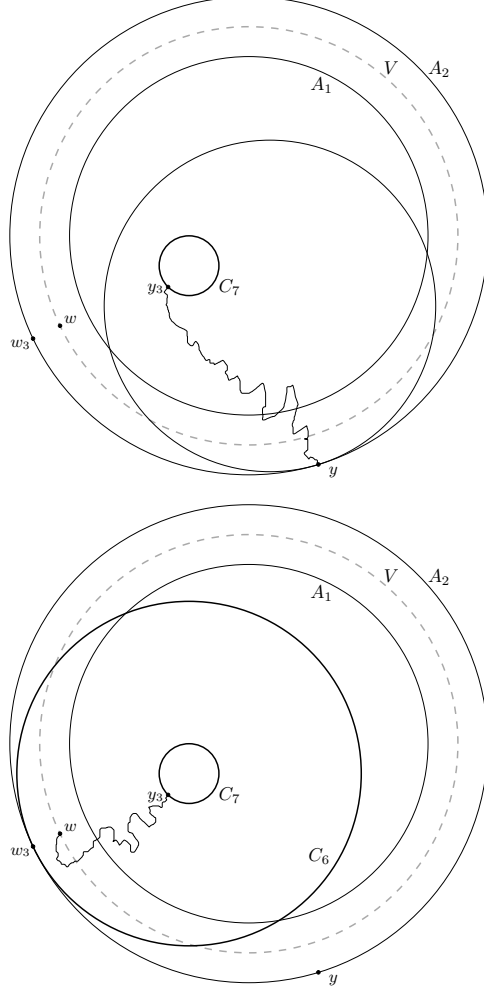


FIGURE 17. We can give a lower bound for $\mathbb{P}_{w_0}[w \xrightarrow{4} y]$ by describing the event where the walk started at w reaches a small sphere C_6 before reaching ∂C_7 and then hits y before any other point in ∂C_8 .

If $h_4 \geq \frac{r}{2}$ we simply replace the balls C_6 and C_8 by A_2^C , the ball C_7 by an discrete ball concentric with A_2^C but with half the diameter, and continue the proof identically.

Let us now provide an upper bound for $\mathbb{P}_{w_0}[w \xrightarrow{4} y]$, which will be needed in the next section. We let C'_6 be a discrete ball of radius $\frac{h_4}{6}$ lying outside A_2^C and intersecting ∂A_2 only at w_3 . We also let C'_7 be a discrete ball of radius $\frac{h_4}{3}$ concentric with C'_6 . Finally we let C'_8 be a discrete ball of radius h_4 lying outside A_2^C and intersecting ∂A_2 only at y .

Then, for the simple random walk started at w to hit ∂A_2 at y , it has first to reach $\partial C_7'$ before hitting $\partial C_6'$ and then hit y before any other point in $\partial C_8'$. As we have already seen, the probability of the first event is of order $\frac{s}{h_4}$ and the probability of the latter is of order $h_4^{-(d-1)}$. This way, we can find a constant $c > 0$ such that:

$$(A.22) \quad \mathbb{P}_{w_0}[w \xrightarrow{4} y] \leq c \frac{s}{h_4} h_4^{-(d-1)}.$$

Finally, using (A.12) and (A.13) we see that the supremum in (A.1) is reached when h_1 and h_3 are of order s . This way, h should have the same order as h_4 . Gathering the bounds (A.12), (A.13), (A.17) and (A.21) we have, for a constant $c > 0$

$$(A.23) \quad \sup_{\substack{w \in V \\ y \in \partial A_2}} \mathbb{P}_w[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y \mid w \xrightarrow{4} y] \leq cs^{-2(d-1)}.$$

We have proved the following proposition:

Proposition A.2. *Regarding the sets A_1°, V° and A_2° , we have that, using the notation defined above, for some constants $c_k > 0$, $k = 1, \dots, 9$, the following bounds are valid:*

$$\begin{aligned} \mathbb{P}_w[w \xrightarrow{1} w_0] &\leq c_1 e^{\frac{-c_2 h_1}{s}} s^{-(d-1)}, \\ \mathbb{P}_w[y \xrightarrow{3} y_0] &\leq c_3 e^{\frac{-c_4 h_3}{s}} s^{-(d-1)} s^{-1}, \\ c_5 \frac{s^2}{h^d} &\leq \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \leq c_6 \frac{s^2}{h^d}, \\ c_7 \frac{s}{h_4^d} &\leq \mathbb{P}_{w_0}[w \xrightarrow{4} y] \leq c_8 \frac{s}{h_4^d}. \\ \sup_{\substack{w \in V \\ y \in \partial A_2}} \mathbb{P}_w[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y \mid w \xrightarrow{4} y] &\leq c_9 s^{-2(d-1)}. \end{aligned}$$

A.1.2. *The hanging probabilities for the smoothed hypercube.* In this subsection we will focus on sets A_1^\square, V^\square and A_2^\square , and the related simple random walk probabilities.

$w \xrightarrow{1} w_0$: We will essentially use the same argument used when the underlying sets were balls. We assume without loss of generality that \mathfrak{H}_{r+2s} is centered at the origin, and let $h_1 := \text{dist}(w_0, y_0)$. We will subdivide the set $A_2^C \setminus A_1$ in sets of diameter of order s in such a way that for a simple random walk trajectory started at w to reach w_0 it will first have to cross a number of order $\frac{h_1}{s}$ of these sets.

Given $j \in \{1, \dots, d\}$, $(m_1, \dots, m_d) \in \{-1, 1\}^d$, $k \in \{1, \dots, j-1, j+1, \dots, d\}$, and $i_k \in \{1, \dots, \lfloor \frac{r}{s} \rfloor\}$, we define

$$E_{j, i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d}^{(m_1, \dots, m_d)} = \left\{ (x_1, \dots, x_d) \in \mathbb{Z}^d; \right. \\ \left. \begin{aligned} x_j &\in [\min\{m_j 2^{-1}r, m_j(2^{-1}r + 2s)\}, \max\{m_j 2^{-1}r, m_j(2^{-1}r + 2s)\}], \\ x_k &\in [m_k(i_k - 1)s, m_k i_k s] \cup [m_k i_k s, m_k(i_k + 1)s]. \end{aligned} \right\}$$

so that there exists a $c_1 > 0$ such that in order for the walk started at w to hit w_0 in A_1 , it will first have to cross at least $\lfloor \frac{c_1 h_1}{s} \rfloor$ sets of the form $E_{j, i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d}^{(m_1, \dots, m_d)}$. Each time the walk reaches a set $E_{j', i'_1, \dots, i'_{j-1}, i'_{j+1}, \dots, i'_d}^{(m'_1, \dots, m'_d)}$, the probability that it will reach another set of the form $E_{j, i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d}^{(m_1, \dots, m_d)}$ at distance at least s from $E_{j', i'_1, \dots, i'_{j-1}, i'_{j+1}, \dots, i'_d}^{(m'_1, \dots, m'_d)}$, before hitting either ∂A_1 or ∂A_2 , is bounded from above by a constant $0 < c_2 < 1$, as can be seen using Donsker's Invariance Principle. Using the strong Markov property, we see that the

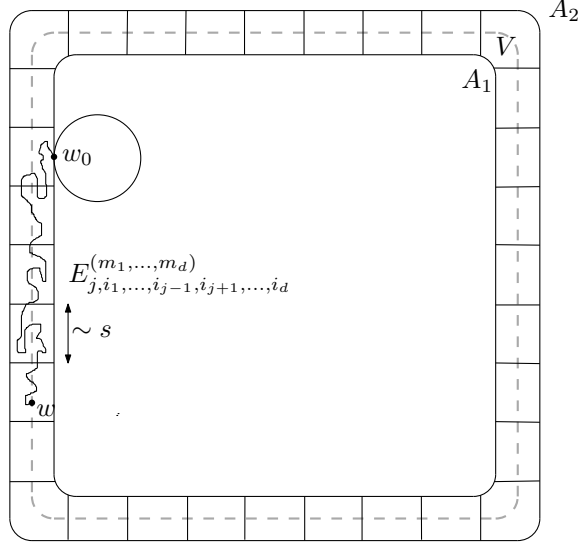


FIGURE 18. A path belonging to $w \xrightarrow{1} w_0$ has to cross $\frac{c_1 h_1}{s}$ sets of the form $E_{j,i_1,\dots,i_{j-1},i_{j+1},\dots,i_d}^{(m_1,\dots,m_d)}$ before hitting w_0 in C'_1 .

probability that the walk started at w crosses $\lfloor \frac{c_1 h_1}{s} \rfloor$ sets of the form $E_{j,i_1,\dots,i_{j-1},i_{j+1},\dots,i_d}^{(m_1,\dots,m_d)}$ is bounded from above by $c_2^{\lfloor \frac{c_1 h_1}{s} \rfloor}$. See Figure 18.

Let C'_1 be a discrete ball of radius s contained in A_1 such that $C'_1 \cap \partial A_1 = w_0$. Recall that the probability that a simple random walk started at a distance of order s from C'_1 will hit C'_1 at w_0 is of order $s^{-(d-1)}$, and that it is harder for a walk started at $x \in A_2^C \setminus A_1$ to first hit A_1 at w_0 than it is for the same walk to first hit C'_1 at w_0 , that is,

$$\mathbb{P}_x[X_{H_{\partial A_1}} = w_0] \leq \mathbb{P}_x[X_{H_{\partial C'_1}} = w_0].$$

In conjunction with last paragraph's argument and the strong Markov property, this shows the existence of a constant $c_3, c_4 > 0$ such that

$$(A.24) \quad \mathbb{P}_w[w \xrightarrow{1} w_0] \leq c_3 e^{\frac{-c_4 h_1}{s}} s^{-(d-1)}.$$

$y_0 \xrightarrow{3} y$: The proof of this bound is essentially the same as that of the corresponding bound in the case when the underlying sets are balls instead of smoothed hypercubes. We have, for some $c_1, c_2 > 0$, and $h_3 := \text{dist}(y, y_0)$,

$$(A.25) \quad \mathbb{P}_w[y \xrightarrow{3} y_0] \leq c_1 e^{\lfloor \frac{-c_2 h_3}{s} \rfloor} s^{-(d-1)} s^{-1}.$$

$w_0 \xrightarrow{2'} y_0$: Let h denote the Euclidean distance between w_0 and y_0 . If $h < 100s$, a simple application of the Green's function bound gives the desired result. We then assume $h > 100s$. Define \tilde{B}_x to be the discrete ball in the ℓ_∞ -norm centered in x with radius $\frac{h}{4\sqrt{d}}$.

We will break up the path $\gamma_2^0 \in w_0 \xrightarrow{2'} y_0$ in pieces that are easier to work with. Let $w_4 \in \partial \tilde{B}_{w_0} \cap A_2^C$, $y_4 \in \partial \tilde{B}_{y_0} \cap A_2^C$. We define the collection of finite paths:

- $w_0 \xrightarrow{5} w_4$: The collection of all finite nearest-neighbor paths starting at w_0 whose only intersection with $\partial \tilde{B}_{w_0} \cup \partial A_2$ is at its ending point $w_4 \in \partial \tilde{B}_{w_0} \cap A_2^C$. It is straightforward to see this collection as a simple random walk event.

- $w_4 \xrightarrow{6} y_4$: The collection of finite nearest-neighbor paths starting at w_4 and ending at y_4 , without intersecting ∂A_2 .
- $y_4 \xrightarrow{7} y_0$: The collection of all finite nearest-neighbor paths that start at y_4 , never return to $\partial \tilde{B}_{y_0} \cap A_2^C$, and end at y_0 without ever reaching ∂A_2 . It is simple to see this collection as a simple random walk event.

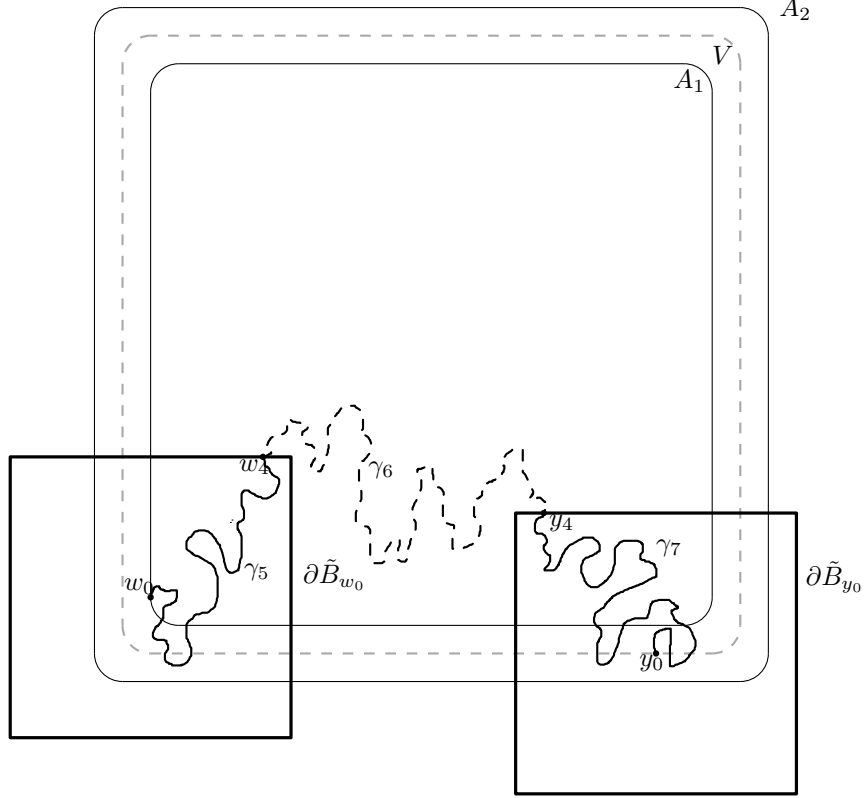


FIGURE 19. Definition of the paths γ_5 , γ_6 and γ_7 .

As before, we denote by $w_0 \xrightarrow{5} w_4 \xrightarrow{6} y_4 \xrightarrow{7} y_0$ the concatenation of these three collections.

Analogously to what we noted at the start of this section, we observe that $\gamma_2^0 \in w_0 \xrightarrow{2'} y_0$ if and only if there exists $w_4 \in \partial \tilde{B}_{w_0} \cap A_2^C$ and $y_4 \in \partial \tilde{B}_{y_0} \cap A_2^C$ such that γ_2^0 is the concatenation of three paths: $\gamma_5 \in w_0 \xrightarrow{5} w_4$, $\gamma_6 \in w_4 \xrightarrow{6} y_4$, and $\gamma_7 \in y_4 \xrightarrow{7} y_0$.

We also define

- $w_4 \xrightarrow{6'} y_4$ The collection of finite simple random walk trajectories starting at w_4 and ending at its first visit to y_4 without intersecting ∂A_2 . This collection can also be seen as the event where the simple random walk started at w_4 visits y_4 before it hits ∂A_2 .

Using the same trick we used to obtain the bound (A.6), we can find a constant $c > 1$ such that

$$(A.26) \quad \mathbb{P}_{w_4}[w_4 \xrightarrow{6'} y_4] \leq \sum_{\gamma_6 \in w_4 \xrightarrow{6} y_4} \frac{1}{2^{|\gamma_6|}} \leq c \mathbb{P}_{w_4}[w_4 \xrightarrow{6} y_4].$$

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We then have

$$\begin{aligned}
\mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] &= \sum_{\substack{w_4 \in \partial \tilde{B}_{w_0} \cap A_2^C \\ \gamma_5 \in w_0 \xrightarrow{5} w_4}} \frac{1}{2^{|\gamma_5|}} \sum_{\substack{y_4 \in \partial \tilde{B}_{y_0} \cap A_2^C \\ \gamma_6 \in w_4 \xrightarrow{6} y_4}} \frac{1}{2^{|\gamma_6|}} \sum_{\gamma_7 \in y_4 \xrightarrow{6} y_0} \frac{1}{2^{|\gamma_7|}} \\
&\leq c \sum_{w_4} \mathbb{P}_{w_0}[w_0 \xrightarrow{5} w_4] \sum_{y_4} \mathbb{P}_{w_4}[w_4 \xrightarrow{6'} y_4] \mathbb{P}_{y_4}[y_4 \xrightarrow{7} y_0].
\end{aligned}
\tag{A.27}$$

We then use the Green's function estimate (2.1) to bound $\mathbb{P}_{w_4}[w_4 \xrightarrow{6'} y_4]$ by ch^{d-2} (note that $\text{dist}(w_4, y_4) = O(h)$). Using this bound on the above equality, we obtain

$$\mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \leq ch^{d-2} \sum_{w_4} \mathbb{P}_{w_0}[w_0 \xrightarrow{5} w_4] \sum_{y_4} \mathbb{P}_{y_4}[y_4 \xrightarrow{7} y_0].
\tag{A.28}$$

We define the events

- $w_0 \xrightarrow{5} \partial \tilde{B}_{w_0}$: The event where the simple random walk started at w_0 reaches $\partial \tilde{B}_{w_0}$ before reaching ∂A_2 .
- $y_0 \xrightarrow{5} \partial \tilde{B}_{y_0}$: The event where the simple random walk started at y_0 reaches $\partial \tilde{B}_{y_0}$ before reaching ∂A_2 .

Note that

$$\sum_{w_4} \mathbb{P}_{w_0}[w_0 \xrightarrow{5} w_4] = \mathbb{P}_{w_0}[w_0 \xrightarrow{5} \partial \tilde{B}_{w_0}],
\tag{A.29}$$

and using the simple random walk's reversibility, we also have

$$\sum_{y_4} \mathbb{P}_{y_4}[y_4 \xrightarrow{5} y_0] = \mathbb{P}_{y_0}[y_0 \xrightarrow{5} \partial \tilde{B}_{y_0}],
\tag{A.30}$$

so that we obtain the following bound

$$\mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \leq \frac{c}{h^{d-2}} \mathbb{P}_{w_0}[w_0 \xrightarrow{5} \partial \tilde{B}_{w_0}] \mathbb{P}_{y_0}[y_0 \xrightarrow{5} \partial \tilde{B}_{y_0}].
\tag{A.31}$$

We still have to obtain a bound for these last two probabilities. Since they are similarly defined, the bound for both of them follows from the same arguments, and thus we will only provide a bound for $\mathbb{P}_{w_0}[w_0 \xrightarrow{5} \partial \tilde{B}_{w_0}]$.

We will do so by looking at the projections of the random walk trajectory in each of the d orthogonal axes. Since we will need to look at these projections independently, we will change our object of study from the simple random walk on \mathbb{Z}^d to the continuous time simple random walk on \mathbb{Z}^d with waiting times between steps distributed as $\text{Exp}(1)$ random variables. Since we will be studying properties of the random walk's trajectories, this change of framework will in no way impact the probabilities of interest. We will denote by \mathbb{P}_x^c , with $x \in \mathbb{Z}^d$, the probability measure associated with such continuous time random walk starting at x .

We recall the definition of \mathfrak{H}_{r+2s} , the unsmoothed version of A_2^C . Here we will assume \mathfrak{H}_{r+2s} takes the form

$$\mathfrak{H}_{r+2s} := \{(x_1, \dots, x_d) \in \mathbb{Z}^d : 0 \leq x_i \leq r + 2s, \text{ for all } i = 1, \dots, d\}$$

Without loss of generality we assume that $0 \in \mathbb{Z}^d$ is the point belonging to $\{0, r + 2s\}^d$ which is closest to w_0 . We denote $w_0 \equiv (w_0^1, \dots, w_0^d)$ and for each $j \in \{1, \dots, d\}$ we let $(X_t^j, t \geq 0)$ be the projection on the j -th axis of the continuous time random walk started at w_0 . This projection is itself a continuous time random walk started at w_0^j

with waiting time between jumps given by a $\text{Exp}(d)$ random variable, and, as we already noted, these random walks are independent from each other. We will define \mathbb{P}_x^j to be the probability measure associated with this projected random walk when it starts at $x \in \mathbb{Z}$.

We define, for $j \in \{1, \dots, d\}$ and $A \subset \mathbb{Z}$, the hitting times

$$(A.32) \quad \tau^j(A) := \inf_{t \geq 0} \{X_t^j \in A\},$$

$$(A.33) \quad \tau^j := \tau^j(\{\max\{0, w_0^j - h\}, w_0^j + h\}),$$

and we let J_t^j denote the number of jumps the continuous time walk projected on the j -th direction makes before time t .

Since J_t^j has Poisson distribution with parameter td^{-1} , we have (using a convenient large deviation estimate):

$$(A.34) \quad \mathbb{P}_{w_0^j}^j \left[\frac{(1-\delta)t}{d} \leq J_t^j \leq \frac{(1+\delta)t}{d} \right] \geq 1 - e^{c(\delta)t}.$$

Given w_0 , we divide the d directions of \mathbb{Z}^d in two kinds. The first kind will be such that $\max\{0, w_0^j - h\} = 0$, the second will be such that $\max\{0, w_0^j - h\} = w_0^j - h$. We assume without loss of generality the first d_0 directions to be of the first kind and the remaining directions to be of the second kind.

Given $t \in \mathbb{R}_+$, we will need to bound the probability $\mathbb{P}_{w_0^j}^j[\tau^j > t]$. We first assume $j \leq d_0$. We denote by $(S_k^x, k \in \mathbb{Z}_+)$ the unidimensional discrete time simple random walk starting at $x \in \mathbb{Z}$, and by $\mathbb{P}_x^{\mathbb{Z}}$ its associated measure. We have

$$(A.35) \quad \mathbb{P}_{w_0^j}^j[\tau^j > t] \leq \mathbb{P}_{w_0^j}^j[\tau_{\{0\}}^j > t] \leq \sum_{t_0} \mathbb{P}_{w_0^j}^{\mathbb{Z}}[\min_{0 \leq k \leq t_0} S_k^{w_0^j} > 0] \mathbb{P}_{w_0^j}^c[J_t^j = t_0].$$

Using (A.34), the reflection principle for the unidimensional simple random walk, and the central limit theorem, we can bound the above expression by

$$(A.36) \quad \begin{aligned} & \sum_{\substack{t_0 \in \mathbb{Z}_+, \\ t_0 \in \left(\frac{(1-\delta)t}{d}, \frac{(1+\delta)t}{d}\right)}} \mathbb{P}_{w_0^j}^c[J_t^j = t_0] \left(1 - 2\mathbb{P}_{w_0^j}^{\mathbb{Z}}[S_{t_0}^{w_0^j} < 0]\right) + e^{-c(\delta)t} \\ &= \sum_{t_0 \in \left(\frac{(1-\delta)t}{d}, \frac{(1+\delta)t}{d}\right)} \mathbb{P}_{w_0^j}^c[J_t^j = t_0] \left(1 - 2\mathbb{P}_0^{\mathbb{Z}}[S_{t_0}^0 > w_0^j]\right) + e^{-c(\delta)t} \end{aligned}$$

$$(A.37) \quad \begin{aligned} & \leq c \sum_{\substack{t_0 \in \mathbb{Z}_+, \\ t_0 \in \left(\frac{(1-\delta)t}{d}, \frac{(1+\delta)t}{d}\right)}} \mathbb{P}_{w_0^j}^c[J_t^j = t_0] \left(1 - \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{v^2}{2}} dv\right) \\ & \quad + \frac{2}{\sqrt{2\pi}} \int_0^{\frac{w_0^j}{\sqrt{t_0}}} e^{-\frac{v^2}{2}} dv + O(\sqrt{t_0}^{-1}) + e^{-c(\delta)t} \\ & \leq c \sum_{\substack{t_0 \in \mathbb{Z}_+, \\ t_0 \in \left(\frac{(1-\delta)t}{d}, \frac{(1+\delta)t}{d}\right)}} \mathbb{P}_{w_0^j}^c[J_t^j = t_0] \frac{w_0^j}{\sqrt{t_0}} \\ & \leq c \frac{w_0^j}{\sqrt{t}}. \end{aligned}$$

In an analogous way, we show for $j > d_0$

$$(A.38) \quad \mathbb{P}_{w_0^j}^j[\tau^j > t] \leq c \frac{h}{\sqrt{t}}.$$

We now bound the probability that the walk exits the sphere $\partial \tilde{B}_{w_0}$ through the first direction, without ever hitting ∂A_2 .

(A.39)

$$\begin{aligned} & \mathbb{P}_{w_0}^c[X_{\tau^1}^1 = w_0^1 + h, \tau^j > \tau^1 \text{ for all } j \neq 1] \\ &= \mathbb{P}_{w_0^1}^1[X_{\tau^1}^1 = w_0^1 + h] \int \prod_{j \neq 1} \mathbb{P}_{w_0^j}^j[\tau^j > t] \mathbb{P}_{w_0}^c[\tau^1 = t + dt \mid X_{\tau^1}^1 = w_0^1 + h], \end{aligned}$$

where $\mathbb{P}_{w_0}^c[\tau^1 = t + dt \mid X_{\tau^1}^1 = w_0^1 + h]$ is the distribution of τ^1 conditioned on the event $\{X_{\tau^1}^1 = w_0^1 + h\}$. Then, using (A.36), (A.38) and the gambler's ruin estimate (see Section 5.1 of [11]), we are able to bound the above expression by

$$(A.40) \quad c \frac{w_0^1}{h} \int \prod_{1 < j \leq d_0} \frac{w_0^j}{\sqrt{t}} \prod_{d_0 < j \leq d} \frac{h}{\sqrt{t}} \mathbb{P}_{w_0}^c[\tau^1 = t + dt \mid X_{\tau^1}^1 = w_0^1 + h].$$

We define the continuous time simple random walk

$$(A.41) \quad X_t^{1, \frac{1}{2}} := X_{t + \tau^1(\{0, [2^{-1}(w_0^1 + h)]\})}^1,$$

so that, on $\{X_{\tau^1}^1 = w_0^1 + h\}$, $X_t^{1, \frac{1}{2}}$ is distributed as a continuous time one-dimensional walk starting at a halfway point between 0 and $w_0^1 + h$. We also define the hitting time

$$(A.42) \quad \tau^{1, \frac{1}{2}} := \inf\{t \geq 0; X_t^{1, \frac{1}{2}} \in \{0, w_0^1 + h\}\}.$$

On the event $\{X_{\tau^1}^1 = w_0^1 + h\}$, τ^1 is distributed as $\tau^1(\{0, [2^{-1}(w_0^1 + h)]\}) + \tau^{1, \frac{1}{2}}$, so that $\{\tau^1 < t\}$ implies $\{\tau^{1, \frac{1}{2}} < t\}$.

We then have, for $\alpha < 1$

$$(A.43) \quad \mathbb{P}_{w_0}^c[\tau^1 < \alpha h^2 d \mid X_{\tau^1}^1 = w_0^1 + h] \leq \mathbb{P}_{w_0}^c[\tau^{1, \frac{1}{2}} < \alpha h^2 d \mid X_{\tau^1}^1 = w_0^1 + h].$$

Since $X_t^{1, \frac{1}{2}}$ starts at a halfway point between 0 and $w_0^1 + h$, we have

$$(A.44) \quad \mathbb{P}_{w_0}^c[\tau^{1, \frac{1}{2}} < \alpha h^2 d \mid X_{\tau^1}^1 = w_0^1 + h] \leq c \mathbb{P}_{2^{-1}(w_0^1 + h)}^1[\tau^{1, \frac{1}{2}} < \alpha h^2 d].$$

Using (A.34) together with a large deviation estimate (see Lemma 1.5.1 of [10]), we obtain

$$(A.45) \quad \mathbb{P}_{w_0}^c[\tau^1 < \alpha h^2 d \mid X_{\tau^1}^1 = w_0^1 + h] \leq e^{c\alpha^{-1}}.$$

We define

$$(A.46) \quad \psi_{w_0}(t) := c \frac{w_0^1}{h} \prod_{1 < j \leq d_0} \frac{w_0^j}{\sqrt{t}} \prod_{d_0 < j \leq d} \frac{h}{\sqrt{t}}.$$

Then

$$\begin{aligned} & \mathbb{P}_{w_0}^c[X_{\tau^1}^1 = w_0^1 + h, \tau^j > \tau^1 \text{ for all } j \neq 1] \\ & \leq c \frac{w_0^1}{h} \int \prod_{1 < j \leq d_0} \frac{w_0^j}{\sqrt{t}} \prod_{d_0 < j \leq d} \frac{h}{\sqrt{t}} \mathbb{P}_{w_0}^c[\tau^1 = t + dt \mid X_{\tau^1}^1 = w_0^1 + h] \end{aligned}$$

$$\begin{aligned}
&= \sum_{k \geq 1} c \frac{w_0^1}{h} \int_{t=h^2 d(k+1)^{-1}}^{t=h^2 dk^{-1}} \prod_{1 < j \leq d_0} \frac{w_0^j}{\sqrt{t}} \prod_{d_0 < j \leq d} \frac{h}{\sqrt{t}} \mathbb{P}_{w_0}^c[\tau^1 = t + dt \mid X_{\tau^1}^1 = w_0^1 + h] \\
&\quad + c \frac{w_0^1}{h} \int_{t \geq h^2 d} \prod_{1 < j \leq d_0} \frac{w_0^j}{\sqrt{t}} \prod_{d_0 < j \leq d} \frac{h}{\sqrt{t}} \mathbb{P}_{w_0}^c[\tau^1 = t + dt \mid X_{\tau^1}^1 = w_0^1 + h] \\
&\leq \psi_{w_0}(h^2 d) + \sum_{k \geq 1} \psi_{w_0}(h^2 dk^{-1}) e^{-ck}.
\end{aligned}$$

Since $\psi_{w_0}(h^2 dk^{-1})$ grows polynomially in k as $k \rightarrow \infty$, we have

$$(A.47) \quad \mathbb{P}_{w_0}^c[X_{\tau^1}^1 = w_0^1 + h, \tau^j > \tau^1 \text{ for all } j \neq 1] \leq c \psi_{w_0}(h^2 d) \leq c' \prod_{1 \leq j \leq d_0} \frac{w_0^j}{h}.$$

The proof is analogous for every $j = 1, \dots, d$. When $j > d_0$ the calculations are in fact easier because, since $\max\{0, w_0^j - h\} = w_0^j - h$, there is no preferential direction in which the random walk $(X_t^j, t \geq 0)$ has to exit the ball $\partial \tilde{B}_{w_0} \cap A_2^C$, so that the required conditioning in (A.39) is simpler. We then have, for j such that $d_0 < j \leq d$,

$$(A.48) \quad \mathbb{P}_{w_0}^c[\tau^n > \tau^j \text{ for all } n \neq j] \leq c \prod_{1 \leq j \leq d_0} \frac{w_0^j}{h},$$

so that

$$\begin{aligned}
(A.49) \quad \mathbb{P}[w_0 \xrightarrow{5} \partial \tilde{B}_{w_0}] &\leq \sum_{1 \leq k \leq d_0} \mathbb{P}_{w_0}^c[X_{\tau^k}^k = w_0^k + h, \tau^n > \tau^k \text{ for all } n \neq j] \\
&\quad + \sum_{d_0 < k \leq d} \mathbb{P}_{w_0}^c[\tau^n > \tau^k \text{ for all } n \neq j] \\
&\leq c \prod_{1 \leq i \leq d_0} \frac{w_0^i}{h}.
\end{aligned}$$

We will change the notation so that we are able to express the inequality above in a way that does not use the fact that $\{0\}^d$ is the vertex of $\{0, r + 2s\}^d$ which is closest to w_0 . Let \mathfrak{H}_i^{d-1} ; $i = 1, \dots, 2d$; denote the $(d-1)$ -dimensional hyperfaces of \mathfrak{H}_{r+2s} , and let $l_i^{w_0} := \min\{\text{dist}(w_0, \mathfrak{H}_i^{d-1}), h\}$, and $l_i^{y_0} := \min\{\text{dist}(y_0, \mathfrak{H}_i^{d-1}), h\}$. Then, (A.49) implies

$$\mathbb{P}[w_0 \xrightarrow{5} \partial \tilde{B}_{w_0}] \leq c \frac{l_1^{w_0} \cdots l_{2d}^{w_0}}{h^{2d}},$$

and using the same arguments used above, we can see that

$$\mathbb{P}[y_0 \xrightarrow{5} \partial \tilde{B}_{y_0}] \leq c \frac{l_1^{y_0} \cdots l_{2d}^{y_0}}{h^{2d}}.$$

Together with (A.31), this shows

$$(A.50) \quad \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \leq ch^{-(d-2)} \frac{l_1^{w_0} \cdots l_{2d}^{w_0}}{h^{2d}} \frac{l_1^{y_0} \cdots l_{2d}^{y_0}}{h^{2d}}.$$

We will also need a matching lower bound. We will continue to use the same notations and conventions. Again we assume $h > 100s$, since otherwise the lower bound follows immediately from using a Green's function estimate. We define

$$(A.51) \quad w_5 := \left(w_0^1 + \frac{h}{4\sqrt{d}}, \dots, w_0^{d_0} + \frac{h}{4\sqrt{d}}, w_0^{d_0+1}, \dots, w_0^d \right),$$

We analogously define y_5 : Let $e_{i_{d_1}}, \dots, e_{i_{d_k}}$ be the vectors in the orthonormal basis of \mathbb{R}^d corresponding to the directions in which the ball $B_\infty(y_0, \frac{h}{4\sqrt{d}})$ passes the limits of the hypercube \mathfrak{H}_{r+2s} . y_5 is defined to be the point in A_2^C such that

$$l = d_1, \dots, d_k \implies |\langle y_5 - y_0, e_{i_l} \rangle| = \frac{h}{4\sqrt{d}},$$

$$n \neq d_1, \dots, d_k \implies |\langle y_5 - y_0, e_{i_n} \rangle| = 0$$

and

$$B_\infty\left(y_5, \frac{h}{4\sqrt{d}}\right) \subseteq \mathfrak{H}_{r+2s}.$$

Our plan is to describe an event contained in $w_0 \xrightarrow{2'} y_0$ with probability matching that of the right side of (A.50). We let

$$B_{w_5} := B_\infty\left(w_5, \frac{h}{16\sqrt{d}}\right),$$

and

$$B_{y_5} := B_\infty\left(y_5, \frac{h}{16\sqrt{d}}\right),$$

For $w_6 \in \partial B_{w_5}$ and $y_6 \in \partial B_{y_5}$, we define the events

- $w_0 \xrightarrow{8} w_6$: The event where the random walk started at w_0 hits ∂B_{w_5} before hitting ∂A_2 and its entrance point in ∂B_{w_5} is w_6 .
- $w_6 \xrightarrow{9} y_6$: The event where the random walk started at w_6 visits $y_6 \in \partial B_{y_5}$ before reaching ∂A_2 .
- $y_6 \xrightarrow{10} y_0$: The event where the simple random walk started at y_6 hits y_0 before returning to ∂B_{y_5} .

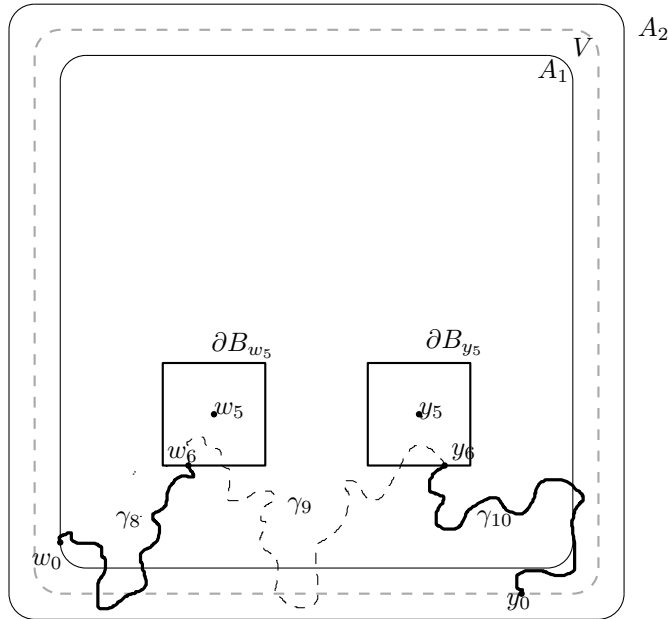


FIGURE 20. Definition of the paths γ_8 , γ_9 and γ_{10} .

And we denote by $w_0 \xrightarrow{8} w_6 \xrightarrow{9} y_6 \xrightarrow{10} y_0$ the “concatenation” of these three events, that is, the path γ belongs to the event $w_0 \xrightarrow{8} w_6 \xrightarrow{9} y_6 \xrightarrow{10} y_0$ if and only if γ is the concatenation of three paths: $\gamma_8 \in w_0 \xrightarrow{8} w_6$, $\gamma_9 \in w_6 \xrightarrow{9} y_6$ and $\gamma_{10} \in y_6 \xrightarrow{10} y_0$. It is then clear that

$$\bigcup_{w_6} \bigcup_{y_6} w_0 \xrightarrow{8} w_6 \xrightarrow{9} y_6 \xrightarrow{10} y_0 \subset w_0 \xrightarrow{2'} y_0,$$

so that; summing over $\gamma_8 \in w_0 \xrightarrow{8} w_6$, $\gamma_9 \in w_6 \xrightarrow{9} y_6$ and $\gamma_{10} \in y_6 \xrightarrow{10} y_0$; we have

$$\begin{aligned} \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] &\geq \sum_{w_6} \sum_{\gamma_8} \frac{1}{2^{|\gamma_8|}} \sum_{y_6} \sum_{\gamma_9} \frac{1}{2^{|\gamma_9|}} \sum_{\gamma_{10}} \frac{1}{2^{|\gamma_{10}|}} \\ (A.52) \quad &= \sum_{w_6} \mathbb{P}_{w_0}[w_0 \xrightarrow{8} w_6] \sum_{y_6} \mathbb{P}_{w_6}[w_6 \xrightarrow{9} y_6] \mathbb{P}_{y_6}[y_6 \xrightarrow{10} y_0] \\ &\geq \frac{c}{h^{d-2}} \sum_{w_6} \mathbb{P}_{w_0}[w_0 \xrightarrow{8} w_6] \sum_{y_6} \mathbb{P}_{y_6}[y_6 \xrightarrow{10} y_0], \end{aligned}$$

where we bounded $\mathbb{P}_{w_6}[w_6 \xrightarrow{9} y_6]$ from below by ch^{d-2} using the Green’s function estimate (2.1) and the fact that the distance of both w_6 and y_6 from ∂A_2 has order h .

We define the events

- $w_0 \xrightarrow{8} \partial B_{w_5}$: The event where the simple random walk started at w_0 reaches ∂B_{w_5} before reaching ∂A_2 .
- $y_0 \xrightarrow{10} \partial B_{y_5}$: The event where the simple random walk started at y_0 reaches ∂B_{y_5} before reaching ∂A_2 .

Due to the simple random walk’s reversibility, we get that

$$(A.53) \quad \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \geq \frac{c}{h^{d-2}} \mathbb{P}_{w_0}[w_0 \xrightarrow{8} \partial B_{w_5}] \mathbb{P}_{y_0}[y_0 \xrightarrow{10} \partial B_{y_5}].$$

We will prove a bound for $\mathbb{P}_{w_0}[w_0 \xrightarrow{8} \partial B_{w_5}]$, since the bound for $\mathbb{P}_{y_0}[y_0 \xrightarrow{10} \partial B_{y_5}]$ follows from analogous arguments. We will use the same continuous time random walk projections to study this probability. The notation used will be the same as the one used in the proof of the upper bound. For $1 \leq j \leq d_0$, we define

$$\tau^j := \inf \left\{ t \geq 0, X_t^j \in \left\{ 0, w_0^j + \frac{h}{4\sqrt{d}} \right\} \right\},$$

and on the event $\{X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}}\}$, we define τ_∞^j as

$$\tau_\infty^j := \inf \left\{ t \geq 0, X_{t+\tau^j}^j \in \left\{ w_0^j + \frac{3h}{16\sqrt{d}}, w_0^j + \frac{5h}{16\sqrt{d}} \right\} \right\},$$

that is, the first time after τ^j when the projection of the continuous time simple random walk on the j -th direction hits the projected boundary of the ball B_{w_5} . For $d_0 < m \leq d$, we also define

$$\tau_\infty^m := \inf \left\{ t \geq 0, X_t^m \in \left\{ w_0^m - \frac{h}{16\sqrt{d}}, w_0^m + \frac{h}{16\sqrt{d}} \right\} \right\},$$

the first time the walk projected in the m -th direction hits the projected boundary of the ball B_{w_5} . Finally, we define

$$T := \max_{1 \leq i \leq d_0} \tau^i.$$

We then have, for any $l > 0$,

$$\begin{aligned}
& \mathbb{P}_{w_0} \left[w_0 \xrightarrow{8} \partial B_{w_5} \right] \\
& \geq \mathbb{P}_{w_0}^c \left[\begin{array}{l} X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}}, \tau_\infty^j > T \text{ for all } j = 1, \dots, d_0 ; \\ \tau_\infty^m > T \text{ for all } m = d_0 + 1, \dots, d \end{array} \right] \\
& \geq \mathbb{P}_{w_0}^c \left[T < l \right] \prod_{1 \leq j \leq d_0} \mathbb{P}_{w_0^j}^j \left[X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}}, \tau_\infty^j > l \right] \prod_{d_0 < m \leq d} \mathbb{P}_{w_0^m}^m \left[\tau_\infty^m > l \right].
\end{aligned}$$

Now, let $c_1, c_2 > 0$ be such that $c_1 > c_2$. We have that

$$\mathbb{P}_{w_0}^c \left[T < c_1 h^2 \right] \geq c > 0.$$

For each $j = 1, \dots, d_0$, we have, by the strong Markov property,

$$\begin{aligned}
\mathbb{P}_{w_0^j}^j \left[X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}}, \tau_\infty^j > c_1 h^2 \right] & \geq \mathbb{P}_{w_0^j}^j \left[X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}}, \tau_\infty^j > c_1 h^2, \tau^j > c_2 h^2 \right] \\
& \geq \mathbb{P}_{w_0^j}^j \left[X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}}, \tau^j > c_2 h^2 \right] \\
& \quad \times \mathbb{P}_{w_0^j + \frac{h}{4\sqrt{d}}}^j \left[H_{\partial B_{w_5}} > (c_1 - c_2) h^2 \right] \\
& \geq c \mathbb{P}_{w_0^j}^j \left[X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}}, \tau^j > c_2 h^2 \right] \\
& \geq c \mathbb{P}_{w_0^j}^j \left[X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}} \right] \mathbb{P}_{w_0^j}^j \left[\tau^j > c_2 h^2 \mid X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}} \right].
\end{aligned}$$

Using (A.45), we can see that

$$\mathbb{P}_{w_0^j}^j \left[\tau^j > c_2 h^2 \mid X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}} \right] > c > 0,$$

so that

$$\begin{aligned}
\mathbb{P}_{w_0^j}^j \left[X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}}, \tau_\infty^j > c_1 h^2 \right] & \geq c \mathbb{P}_{w_0^j}^j \left[X_{\tau^j}^j = w_0^j + \frac{h}{4\sqrt{d}} \right] \\
& \geq c \frac{w_0^j}{h}.
\end{aligned}$$

For each $m = d_0 + 1, \dots, d$, it is elementary to see that

$$\mathbb{P}_{w_0^m}^m \left[\tau_\infty^m > c_1 h^2 \right] \geq c > 0.$$

Collecting the above equations, we obtain that

$$(A.54) \quad \mathbb{P}_{w_0} \left[w_0 \xrightarrow{8} \partial B_{w_5} \right] \geq \prod_{1 \leq i \leq d_0} \frac{w_0^i}{h}.$$

Together with (A.53) and using the new notation, we have established the bounds:

$$(A.55) \quad ch^{-(d-2)} \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h^{2d}} \cdot \frac{l_1^{y_0} \dots l_{2d}^{y_0}}{h^{2d}} \leq \mathbb{P}_{w_0} \left[w_0 \xrightarrow{2'} y_0 \right] \leq c' h^{-(d-2)} \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h^{2d}} \cdot \frac{l_1^{y_0} \dots l_{2d}^{y_0}}{h^{2d}}.$$

$w \xrightarrow{4} y$: Again we let $h_4 := \text{dist}(w, y)$, and again we suppose $h_4 > 100s$, since a elementary application of the estimate for the Green's function proves the case when $h_4 < 100s$. We will start with the lower bound. Let C_3 be a discrete ball of radius s contained in A_2^C such that $\partial A_2 \cap C_3 = \{y\}$. Let C_4 be a discrete ball of radius $\frac{s}{4}$ concentric with C_3 . Then, the probability that the walk started at y hits C_4 before returning to ∂A_2

is bigger than the probability that it hits C_4 before returning to C_3 , and has order s^{-1} . Now, for every point $\tilde{y} \in \partial C_4$, we bound $\mathbb{P}_w[w \xrightarrow{2'} \tilde{y}]$ from below in exactly the same way as we bounded $\mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0]$. So that, using the walk's reversibility, the fact that $h_4 > 100s$, and the same notation introduced above, we have

$$\begin{aligned} \mathbb{P}_w[w \xrightarrow{4} y] &\geq \sum_{\tilde{y} \in \partial C_4} \mathbb{P}_w[w \xrightarrow{2'} \tilde{y}] \mathbb{P}_y[H_{\partial C_4} < H_{\partial A_2}, X_{H_{\partial C_4}} = \tilde{y}] \\ &\geq cs^{-1} \inf_{\tilde{y} \in \partial C_4} \mathbb{P}_w[w \xrightarrow{2'} \tilde{y}] \\ &\geq cs^{-1} h_4^{-(d-2)} \inf_{\tilde{y} \in \partial C_4} \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h_4^{2d}} \cdot \frac{l_1^{\tilde{y}} \dots l_{2d}^{\tilde{y}}}{h_4^{2d}}. \end{aligned}$$

For the upper bound, let C'_3 be a discrete ball of radius s contained in $A_2 \cup \partial A_2$ such that $\partial A_2 \cap C'_3 = \{y\}$. Let C'_4 be a discrete ball of radius $2s$ concentric with C'_3 . Then

$$\begin{aligned} \mathbb{P}_w[w \xrightarrow{4} y] &\leq \sum_{\hat{y} \in \partial C'_4} \mathbb{P}_w[w \xrightarrow{2'} \hat{y}] \mathbb{P}_y[H_{\partial C'_4} < H_{\partial A_2}, X_{H_{\partial C'_4}} = \hat{y}] \\ &\leq cs^{-1} \inf_{\hat{y} \in \partial C'_4} \mathbb{P}_w[w \xrightarrow{2'} \hat{y}] \\ &\leq cs^{-1} h_4^{-(d-2)} \sup_{\hat{y} \in \partial C'_4} \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h_4^{2d}} \frac{l_1^{\hat{y}} \dots l_{2d}^{\hat{y}}}{h_4^{2d}}. \end{aligned}$$

Using (A.24) and (A.25) we see that the supremum in (A.1) is reached when h_1 and h_3 are of order s . This way, h should have the same order as h_4 . We have proved the following proposition:

Proposition A.3. *Regarding the sets A_1^\square, V^\square and A_2^\square , we have that, using the notation defined above, for some constants $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 > 0$, the following bounds are valid:*

$$\begin{aligned} \mathbb{P}_w[w \xrightarrow{1} w_0] &\leq c_1 \exp\left(\frac{-c_2 h_1}{s}\right) s^{-(d-1)}, \\ \mathbb{P}_w[y \xrightarrow{3} y_0] &\leq c_3 \exp\left(\frac{-c_4 h_3}{s}\right) s^{-(d-1)} s^{-1}, \\ c_5 h^{-(d-2)} \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h^{2d}} \frac{l_1^{y_0} \dots l_{2d}^{y_0}}{h^{2d}} &\leq \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \leq c_6 h^{-(d-2)} \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h^{2d}} \frac{l_1^{y_0} \dots l_{2d}^{y_0}}{h^{2d}}, \\ c_7 s^{-1} h_4^{-(d-2)} \inf_{\tilde{y} \in \partial C_4} \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h_4^{2d}} \frac{l_1^{\tilde{y}} \dots l_{2d}^{\tilde{y}}}{h_4^{2d}} &\leq \mathbb{P}_{w_0}[w \xrightarrow{4} y] \leq c_8 s^{-1} h_4^{-(d-2)} \sup_{\hat{y} \in \partial C'_4} \frac{l_1^{w_0} \dots l_{2d}^{w_0}}{h_4^{2d}} \frac{l_1^{\hat{y}} \dots l_{2d}^{\hat{y}}}{h_4^{2d}}, \\ \sup_{\substack{w \in V \\ y \in \partial A_2}} \mathbb{P}_w[w \xrightarrow{1} w_0 \xrightarrow{2} y_0 \xrightarrow{3} y \mid w \xrightarrow{4} y] &\leq c_9 s^{-2(d-1)}. \end{aligned}$$

A.2. Proof of Lemma 5.1. Let $z \in \Sigma$ be such that $\Xi(z) = (w_0, y_0)$, and again let h stand for the Euclidean distance between w_0 and y_0 . We let $\pi(w_0, y_0)$ be defined in the same way as in (5.1). Given a simple random walk trajectory ϱ started in a set B containing V , we define $\mathcal{C}_{w_0, y_0}^B(\varrho)$ to be the function that counts how many times the random walk trajectory ϱ makes an excursion on A_2^C that enters A_1 at w_0 , and y_0 is the last point such excursion visits on V before reaching ∂A_2 . We let \mathcal{C}_{w_0, y_0}^B denote the random variable $\mathcal{C}_{w_0, y_0}^B(\bar{\varrho})$ when $\bar{\varrho}$'s first point is chosen according to \bar{e}_B . Proposition 4.1 then implies

$$\pi(w_0, y_0) = \mathbb{E}(\mathcal{C}_{w_0, y_0}^V).$$

Define $\tilde{V} := \partial B(0, 3(r+s))$, the discrete sphere of radius $3(r+s)$. We define

$$\tilde{\pi}(w_0, y_0) := \mathbb{E}(\mathcal{C}_{w_0, y_0}^{\tilde{V}}).$$

From the compatibility of the laws defined in (2.2), one can see that (see also the proof of Lemma 6.2 of [13]):

$$u \operatorname{cap}(\tilde{V}) \mathbb{E}(\mathcal{C}_{w_0, y_0}^{\tilde{V}}) = u \operatorname{cap}(V) \mathbb{E}(\mathcal{C}_{w_0, y_0}^V).$$

Since $\operatorname{cap}(\tilde{V}) \asymp \operatorname{cap}(V)$, if we successfully estimate $\tilde{\pi}(w_0, y_0)$ we will automatically be provided with an estimate for $\pi(w_0, y_0)$. We changed the problem from estimating $\pi(w_0, y_0)$ to estimating $\tilde{\pi}(w_0, y_0)$ so that the distance between the simple random walk's starting point and w_0 does not affect our calculations.

First we note that $\mathcal{C}_{w_0, y_0}^{\tilde{V}}$ is dominated by a Geometric (c_1) random variable, for some $0 < c_1 < 1$. This follows from the fact that every time the simple random walk exits A_2^C , with probability uniformly greater than some constant $1 - c_1 > 0$, the walk never returns to w_0 . This way, it will be sufficient to estimate the probability $\mathcal{P}[\mathcal{C}_{w_0, y_0}^{\tilde{V}} \geq 1]$ for our purposes.

So, for a walk started at \tilde{V} to reach w_0 , it first has to hit a discrete sphere ∂C_1 of radius $\frac{s}{2}$ centered on w_0 . The probability of such event is of order $\frac{s^{d-2}}{r^{d-2}}$, by Proposition 6.4.2 of [11].

Let C_2 be a discrete ball of radius s contained in A_1 such that $C_2 \cap A_1 = \{w_0\}$. We also let C_3 be a discrete ball of radius $2s$ lying outside A_1 such that $C_3 \cap A_1 = \{w_0\}$. Using Proposition 6.5.4 of [11] we have, for any $x' \in \partial C_1 \cap A_1^C$ and some constant $c_2 > 0$:

$$\mathbb{P}_{x'}[X_{H_{A_1}} = w_0] \leq \mathbb{P}_{x'}[X_{H_{C_2}} = w_0] \leq c_2 s^{-(d-1)}.$$

Then, recalling the notation $f_{A_1}(w_0, y_0) := \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0]$ and the fact that $\operatorname{cap}(V) \asymp r^{(d-2)}$, and using the strong Markov property, we get, for constants $c, c_1 > 0$:

$$(A.56) \quad \pi(w_0, y_0) \leq c \mathcal{P}[\mathcal{C}_{w_0, y_0}^{\tilde{V}} \geq 1] \leq c_1 \operatorname{cap}(V)^{-1} s^{-1} f_{A_1}(w_0, y_0).$$

For the lower bound, we let C_4 be a discrete ball of radius $\frac{s}{4}$ contained in $A_2^C \setminus B(0, r+s)$ such that for every $x \in C_4$, $\operatorname{dist}(x, w_0) \leq 2s$. Using the strong Markov property, we get

$$\mathcal{P}[\mathcal{C}_{w_0, y_0}^{\tilde{V}} \geq 1] \geq \inf_{x \in \tilde{V}} \mathbb{P}_x[H_{C_4} < \infty] \inf_{x'' \in C_4} \mathbb{P}_{x''}[X_{C_3} = w_0] f_{A_1}(w_0, y_0),$$

so that, using Proposition 6.4.2 of [11] we have, for some constant $c_3 > 0$,

$$\pi(w_0, y_0) \geq c_3 \operatorname{cap}(V)^{-1} s^{-1} f_{A_1}(w_0, y_0).$$

The part (ii) then follows from (i) and Proposition 4.2.

A.3. A lower bound for α . Let $z \in \Sigma$ be such that $\Xi(z) = (w_0, y_0)$, let $c_4 > 0$ be some positive real number. For

$$\Gamma_{w_0, y_0} := \{(w'_0, y'_0) \in V \times \partial A_2; \max\{\|w'_0 - w_0\|, \|y'_0 - y_0\|\} \leq c_4 s\}$$

and

$$\alpha := \inf \left\{ \frac{g_{(w, y)}(z')}{g_{(w, y)}(\hat{z})}; (w, y) \in V \times \partial A_2, z' \in \Gamma_{w_0, y_0}, \hat{z} \in \mathcal{K} \right\}.$$

We need to find a constant lower bound for α . Such lower bound will be provided if we bound the ratios:

$$(A.57) \quad \inf_{\|w'_0 - w_0\| \leq c_4 s} \frac{\mathbb{P}_w[w \xrightarrow{1} w'_0]}{\mathbb{P}_w[w \xrightarrow{1} w_0]}, \quad \inf_{\|y'_0 - y_0\| \leq c_4 s} \frac{\mathbb{P}_y[y \xrightarrow{3'} y'_0]}{\mathbb{P}_y[y \xrightarrow{3'} y_0]}$$

as the other terms of the product

$$\mathbb{P}_w[w \xrightarrow{1} w_0] \mathbb{P}_{w_0}[w_0 \xrightarrow{2'} y_0] \mathbb{P}_y[y \xrightarrow{3'} y_0] \mathbb{P}_w[w \xrightarrow{4} y]^{-1} = g_{(w,y)}(z)$$

already have matching lower and upper bounds. Since the ratios in (A.57) are very similarly defined, we will only give a lower bound to the first one. We define:

$$D = \left\{ x \in \mathbb{Z}^d \setminus A_1 : \text{dist}(x, A_1) \leq \frac{s}{8} \text{ and } \max\{\text{dist}(x, w_0) \text{dist}(x, w'_0)\} \leq c_4 s \right\},$$

and

$$\hat{D} = \{x \in D : \text{there exists } v \in \mathbb{Z}^d \setminus (A_1 \cup D) \text{ such that } x \leftrightarrow v\}.$$

One can think of \hat{D} as the part of the internal boundary of D that is not adjacent to A_1 .

Proposition 8.7 of [13] then says

$$(A.58) \quad \inf_{\substack{x \in \hat{D} \\ \mathbb{P}_x[X_{H_{A_1}} = w'_0] > 0}} \frac{\mathbb{P}_x[X_{H_{A_1}} = w_0]}{\mathbb{P}_x[X_{H_{A_1}} = w'_0]} > c_4 > 0.$$

Informally the above inequality says that if a random walk is sufficiently away from the points w_0 and w'_0 , but somewhat close to ∂A_1 , then the probabilities that such walk hits either w_0 or w'_0 are comparable.

Changing the notation, we have

$$(A.59) \quad \frac{\mathbb{P}_w[w \xrightarrow{1} w_0]}{\mathbb{P}_w[w \xrightarrow{1} w'_0]} = \frac{\mathbb{P}_w[X_{H_{A_1 \cup \partial A_2}} = w_0]}{\mathbb{P}_w[X_{H_{A_1 \cup \partial A_2}} = w'_0]}.$$

Using the strong Markov property we can rewrite the above ratio between probabilities as the ratio between the sums:

$$(A.60) \quad \frac{\mathbb{P}_w[X_{H_{A_1 \cup \partial A_2}} = w_0]}{\mathbb{P}_w[X_{H_{A_1 \cup \partial A_2}} = w'_0]} = \frac{\sum_{x \in \hat{D}} \mathbb{P}_w[X_{H_{\hat{D} \cup \partial A_2 \cup A_1}} = x] \mathbb{P}_x[X_{H_{A_1 \cup \partial A_2}} = w_0]}{\sum_{x \in \hat{D}} \mathbb{P}_w[X_{H_{\hat{D} \cup \partial A_2 \cup A_1}} = x] \mathbb{P}_x[X_{H_{A_1 \cup \partial A_2}} = w'_0]}.$$

But at the same time

$$\mathbb{P}_x[X_{H_{A_1 \cup \partial A_2}} = w_0] \geq \mathbb{P}_x[X_{H_{A_1}} = w_0] - \mathbb{P}_x[H_{\partial A_2} < H_{A_1}] \sup_{x' \in \partial A_2} \mathbb{P}_{x'}[X_{H_{A_1}} = w_0].$$

By the usual trick of considering the probabilities of hitting and escaping certain well placed discrete balls, we are able to see that both terms in the right side of the inequality have order $\text{dist}(x, \partial A_1) s^{-1} s^{-(d-1)}$. We can then fine-tune the constant c_4 in the definition of Γ_{w_0, y_0} in such a way that

$$\mathbb{P}_x[X_{H_{A_1 \cup \partial A_2}} = w_0] \geq c \mathbb{P}_x[X_{H_{A_1}} = w_0],$$

for some constant $c > 0$. The same is valid for w'_0 , so that

$$\frac{\sum_{x \in \hat{D}} \mathbb{P}_w[X_{H_{\hat{D} \cup \partial A_2 \cup A_1}} = x] \mathbb{P}_x[X_{H_{A_1 \cup \partial A_2}} = w_0]}{\sum_{x \in \hat{D}} \mathbb{P}_w[X_{H_{\hat{D} \cup \partial A_2 \cup A_1}} = x] \mathbb{P}_x[X_{H_{A_1 \cup \partial A_2}} = w'_0]} \geq c \frac{\sum_{x \in \hat{D}} \mathbb{P}_w[X_{H_{\hat{D} \cup \partial A_2 \cup A_1}} = x] \mathbb{P}_x[X_{H_{A_1}} = w_0]}{\sum_{x \in \hat{D}} \mathbb{P}_w[X_{H_{\hat{D} \cup \partial A_2 \cup A_1}} = x] \mathbb{P}_x[X_{H_{A_1}} = w'_0]}.$$

Using (A.58) again we obtain

$$(A.61) \quad \inf_{w'_0 : \|w'_0 - w_0\|} \frac{\mathbb{P}_w[w \xrightarrow{1} w_0]}{\mathbb{P}_w[w \xrightarrow{1} w'_0]} \geq c_2 > 0.$$

This fact together with the arguments presented above show the existence of a constant $c > 0$ such that $\alpha \geq c$, which concludes the proof of the uniform lower bound.

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